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On a Generalized Semicommutative Ring

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ABSTRACT

Let R be a ring with identity. In this paper, we introduce some new results in a class of rings which refers to generalization of semicommutative rings called Q-semicommutative rings whenever $x^2=0$ implies xRx=0, for any a ∈R [7]. They study investigates general properties of Q-semicommutative rings and shows several results of semicommutative ring can be extended to Q-semicommutative rings .

1- Introduction

All rings in this research are associative ring with identity unless we have another state . It is better to give a ring R and used $N^*(R)$ and N(R) to denote the nilradical (the sum of all nil ideal), the set of all nilpotent element is in R respectively . According to H.E. Bell [1], a ring R is called the Insertion of Factor property (IFP) if $xy = 0$ implies $xRy = 0$, for x,y ∈R. Shin[2] used the terms semicommutative and SI for IFP. And Habeb[3]used the term zero-insertive (simple ZI) for IFP. According to Mark [4], R is called NI if $N^*(R)$ = N(R), and a ring R is called 2-primal if it is prime radical which coincides with the set of nilpotent element of the ring (i.e. P(R) = N(R)) and a prime radical P(R) of a ring R is the intersection of all prime ideal of R. In [5] Ham, a ring is called abelian if every idempotent is central. It is clear that every commutative rings are semicommutative rings. According to Cohn [6], a ring R is called reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$. In this paper, we will introduce our main concept namely Q-semicommutative ring which is generalization of semicommutative ring . For several years, the applications of semicommutative rings have been studied by many authors. Kim and Lee in [3] "show that if R is a reduced (a ring is reduced if it has zero element),then $S_3(R)$ in (proposition 2.15) is a semicommutative ring . The study also focuses on various investigated various properties of these ring and their relationships with our known rings .

2- Q-semicommutative ring

Before introducing a new kind of a rings, it is mentioned that this kind of rings is called Q-semicommutative ring. It is significant to state some definitions, propositions and lemmas which will be used later to achieve our main target .

Definition 2.1 A ring R is called semicommutative ring(simply SC) if we need any x ,y ∈ R, xy=0 implies xRy =0 . [7]

Definition 2.2 A commutative ring is said to be a reduced ring if it has no – non zero nilpotent element. [8]

Definition 2.3 A ring R is said to be semiprime if P(R) = 0 . [9]

Definition 2.4 A proper ideal P of a ring R is semiprime ideal if R/P is semiprime ideal. [10]

Definition 2.5 A ring R is called homomorphically semicommutative(simply HSC) if R/I is semicommutative for every proper ideal I in R . [11]

Definition 2.6 A prime ideal I of a ring R is called completely prime if R/I is a domain. i.e. if for a,b ∈R, ab ∈I implies a∈ I or b∈I . [12]

Definition 2.7 Any ideal I of a ring R is said to be completely semiprime if R/I is a reduced ring. i.e if for x ∈R, a^2 ∈I implies a ∈I . [13]

Definition 2.8 A ring R is called a strong 2-primal ring if P(R\I) = N(R\I) for all proper Ideal I of R, where the term proper means only I≠R . [14]

Definition 2.9 An element x of a ring R is regular (in the sense of Von Neumann) if there exists a ∈ R such that xax = x . [9]

Definition 2.10 A ring R is called a left (right) duo ring if every left(right) ideal is two-sided . [15]

Definition 2.11 A ring is said to be Q-semicommutative ring (simply QSC) if $x^2 = 0$ implies xRx = 0 .[7]

Note 2.12 : It is clear that all SC rings are QSC ring , but in general the converse is not true for example **.**

Example 2.13 : Let R = $\begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, where F is division ring.Then R is QSC but it is not SC ring .

Proof : Let β = $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ ∈ R , suppose $\beta^2 = 0$, then a = c = 0 .

So βRβ = $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, for all $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ ∈ R .

Hence R is QSC ring .

Now if $y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\alpha = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $y\alpha = 0$

But $y\alpha \neq 0$. So R is not SC ring .

Let's recall the definition idempotent .

An element e ∈R it is repeated there times to be idempotent if e^2=e .[14]

Note 2.14 : We see that all SC rings are abelian, but in QSC rings it is not true**, In example (2.13)**, we see that $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is an idempotent in R = $\begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, but it is not central .

Proposition 2.15 [3]: Let R be a reduced ring . Then

S = $\left\{\begin{pmatrix} x & y & z \\ 0 & x & w \\ 0 & 0 & x \end{pmatrix} \middle| x, y, z, w \in R\right\}$ is a SC ring .

Theorem 2.16: Let R be a ring . Suppose that R/I is a SC ring for some ideal I of R . If I is reduced, then R is SC . (I here is considered as a ring with identity)

Proof : Let xy= 0 ∈ I Then xRx ⊆ I since R/I is SC . Also $(yIx)^2$ = yI(xy)Ix = 0. So yIx =0 as I is reduced and yIx ⊆ I.

Hence $((xRy)I)^2$ = xRyI xRyI =xR(yIx)RyI =0 ,Hence (xRy)I =0 as (xRy)I ⊆ I and I are reduced . So $(xRy)^2$ ⊆ (xRy)I= 0 which implies

xRy = 0 .As xRy ⊆ I and I is reduced. Therefore, R is SC ring .

Cohn [6] proved the following Theorem .

Let's recall the definition a domain .

A ring R is called a domain if xy=0 in R ,then x=0 or y=0 .[10]

Theorem 2.17[6] : Let R be a ring, then

1- R is a prime and reversible if and only if R is a domain .

2- R is a semiprime and reversible if and only if R is reduced .

Now it is significant to give special attention to the following Theorem.

Theorem 2.18: Let R be a ring, then :

1- R is prime and SC if and only if R is a domain .

2- R is semiprime and SC if and only if R is reduced .

Proof :(1). Suppose that R is prime and SC to prove that R is domain .

Let xy=0, x,y∈R .Then xRy=0, as R is SC .

So x=0 or y=0 since R is prime. Therefore, R is domain .

Conversely : Assume that R is domain and to prove R is prime and

SC. If xRy =0, then xy=0 .

So x=0 or y=0 as R is domain . Thus R is prime .

Let xy=0, then x=0 or y=0 as R is domain .

In either case xRy=0. So R is SC .

(**2**). Suppose that R is semiprime and SC to prove which is reduced .

Let x^2=0, then xRx=0 as R is SC .

Thus x=0 as R is semiprime, so R is reduced .

Conversely : Assume that R is reduced and to prove R is semiprime and SC .

Let xRx =0, Then x^2=0, so x=0, as R is reduced.

Therefore, R is semiprime .

So R is SC by proposition **(2.15)** .

Definition 2.19: A sequence $a_0, a_1, \ldots$ in a ring R is called an m-sequence

if $a_{k+1} \in a_k R a_k$ for each k≥0 . [9]

Lemma 2.20: If a sequence $x_0, x_1, \ldots$ is an m-sequence then A={$x_0, x_1, \ldots$} is m-sequence . [9]

Proof: Let x_m, x_n ∈A, we must show that ∃ some r∈ R such that $x_m r x$ ∈A.

We can assume m≥n without loss of generality

We have $x_{n+1} \in x_n R x_n$.

So $x_{n+2} \in x_{n+1} R x_{n+1} \subseteq (x_n R x_n) R x_{n+1} \subseteq x_n R x_{n+1}$.

Again $x_{n+3} \in x_{n+2} R x_{n+2} \subset (x_n R x_{n+1}) R x_{n+2} \subset x_n R x_{n+2}$.

Containing we have $x_{n+k} \in x_n R x_{n+k-1}$,∀ k≥1

Taking k=m-n+1, then we have $x_{m+1} \in x_n R x_m$, and x_{m+1}∈A .

So ∃ some r∈ R such that $x_m r x_n$ ∈A . thus A is m-system .

Definition 2.21 : The nilradical $N^*(R)$ of the ring R is the sum of all nil ideals of R, which is the largest nil ideal in R . **[16]**

Definition 2.22: An element x ∈R is called **strongly nilpotent** if for any m-sequence $x_0, x_1, \ldots$ with x_0=x, ∃ some n such that x_n=0 . [11]

Lemma 2.23: Every element in the prime radical P(R) of a ring R is strongly nilpotent. Hence P(R) is a nil ideal, so for every R we have

$P(R) \subseteq N^*(R) \subseteq N(R)$. [11]

Lemma 2.24: Let R be a ring then .

1- P(R) = N(R) ⇔ R/P(R) is reduced .

2- $N^*(R)$ = N(R) ⇔N(R) is an ideal of R .

Proof (1): P(R) ⊆N(R), it is held by **Lemma(2.23)** with equality because R/P is reduced ⇔N(R)⊆P(R) .

Proof (2): The nilradical of a ring R is the largest nil ideal by **Definition (2.21)** and $N^*(R)$ ⊆ N(R), so the result follows because N(R) is an ideal if and only if N(R) ⊆ $N^*(R)$.

Theorem 2.25: Let R be a ring: reduced ⇒ reversible ⇒ SC ⇒ abelian . [17]

Proof : Let R be reduced and let xy=0 in R .
Then $(xy)^2$ =yxyx= y(xy)x = 0
So yx =0 since R has no nonzero nilpotent. Hence R is reversible .
Let R be reversible and let xy =0 in R. Then yx =0 and
y(xr) = (yx)r = 0 for any r ∈ R. So xry = 0 as R is reversible
Hence R is SC .
Suppose R is SC . Let $0\neq e=e^2 \in R$.
Then $e(1-e)=e-e^2 =0$
Therefore, eR(1-e) = (1-e)Re =0
Since R is SC .So er(1-e)=0 =(1-e)re for each r∈ R .
Therefore, e is central .
Hence R is abelian .
Note that the implications of above are not true in general, .We see the example (1.10) in [3]

Theorem 2.26: Any left or right duo ring is HSC ring . [16]

Proof : Let R be a left duo ring. Suppose P is any ideal of R,
let xy ∈P, then xR⊆Rx since R is left duo, and so
xRy⊆Rxy⊆P , this shows that R/P is SC.
So R is HSC .
(The right case is similarly proved) .

Lemma 2.27: If for each x∈N(R), $(RxR)^m =0$ for some positive integer m, then R is 2-primal

Proof : Let x∈N(R), $(RxR)^m =0\subseteq P(R) \Longrightarrow RxR \subseteq P(R)$ as P(R) is semiprime. Therefore, x∈P(R) and N(P)⊆P(R) since P(R) is nil ideal of R, we have P(R)⊆N(R). Hence P(R)=N(R) .
So R is 2-primal .

Theorem 2.28: Every semicommutative ring R is 2-primal .

Proof: Let x∈N(R) suppose $x^n=0$, for some positive integer n .
Since R is SC, we have $xRx^{n-1} =0$.
And so $xRxRx^{n-2} =0$, continue by inductively, we get xRxR…..Rx =0,
Hence $(RxR)^n =0$ by **Lemma (2.27)** therefore, R is 2-primal . Now if R is a Von Neumann regular ring, we get the following Theorem .

Theorem 2.29: Let R be a Von Neumann regular ring . The following are equivalent :

1- R is abelian.
2- R is right(left).
3- R is reduced.
4- R is reversible.
5- R is SC ring.
6- R is 2-primal ring.
7- R is NI ring.
8- R is HSC ring.

Proof : (1)→(2) Every principle right (left) ideal of R is generated by a Central Idempotent for a von Neumann regular ring R. Therefore all right(left) ideals are two sided ideal .
(2)→(3) let $r^2=0$. Then rR is a right ideal of R. There for is two sided by (2). So we have RrR ⊆ rR . Since R is regular ∃ a ∈R such that r = r×r and ar ∈ RrR ⊆ rR .
So ∃ some y ∈R such that ar = rb .
Therefore $r =r\times r = rrb = r^2b = 0$. So R is reduced .
{In fact, every regular ring is semiprime and the right duo rings are SC ring}
Apply theorem (2.18) to get (2)→(3)
(3)→(4)→(5) is by theorem (2.25)
(5)→(6) by theorem (2.28) is clear .
(7)→(1) let e be an idempotent in R.
We have er(1-e) when is nilpotent, for any r∈R .
∵ R is regular, ∃ e∈R such that er(1-e)=(er(1-e))z(er(1-e)) .
So er(1-e)z is an idempotent .
But er(1-e)z is also nilpotent as R is NI(i.e N(R)∆R).
So it is 0. Hence we have er(1-e)=0 so er=ere .
Similarly re =ere .
Thus we have er=re, for any r ∈R .
It follows that R is abelian
∴ (1)→(7) is equal .
(8)→(5) is clear and (2)→(8) by theorem (2.26) .

Theorem 2.30: Let R be a ring, suppose R/I is QSC ring, and I is reduced(where I is considered to be a ring without identity). Then R is QSC ring .

Proof : Let $x^2 =0\in I$, where x ∈R. Then xRx ⊆I as R/I is QSC ring and $(xRx)(xRx)=xRx^2Rx=0$. So xRx = 0 as I is reduced. Hence R is QSC ring .

Note 2.31

It is clear that the above theorem is the analog of **Theorem (2.16)** .

Corollary 2.32: Let S be a commutative subring of R if I is a reduced ideal of R. Then S+I is a QSC ring.

Proof : We see that I is also a reduced ideal of S+I .
Since (S+I)/I ≅ S/(S+I) and S is commutative .
We have S+I/I is commutative . Therefore, S+I/I is QSC ring By
Theorem (2.30) . We get S+I which is QSC ring .

Corollary 2.33: Let S be a QSC subring of a ring R and let I be a reduced ideal of R such that S∩I =0, then S+I is QSC ring .

Proof : we see that I is also reduced ideal of S+I .
Since (S+I)/I ≅ S/(S∩I) and S∩I =0. We have (S+I)/I which is QSC ring. **By Theorem (2.30)**, we get S+I which is QSC ring .
Now we show that some Theorems of SC ring are also QSC ring .

Theorem 2.34: Let R be QSC ring and semiprime if and only if it is reduced .

Proof : ⟹ Assume $x^2 =0$, x ∈R
Then xRx = 0 as R is QSC ring . So we have x=0 since R is semiprime. Hence R is reduced .
Conversely: Let $x^2 =0$, then x=0 as R is reduced .
Therefore, xRx=0, R is QSC ring .Now let xRx =0 as 1∈R .
This implies that x=0 as R is reduced . Therefore, R is semiprime.

Theorem 2.35: A ring R is QSC ring and prime if and only if it is a domain .

Proof : Assume xy=0, then $(yx)^2 = y(xy)x = 0$.
It has become clear that every prime ring is semiprime, focusing on R which is reduced by **Theorem (2.33)** Therefore, yx =0 .
And (xRy)(xRy) = xR(yx)Ry =0 . So xRy =0, as R is reduced . This implies that x=0 or y=0, as R is prime .
Hence R is a domain .
Conversely : Let xRy =0, then xy =0 as 1∈R . So x =0 or y=0 as R is a
Domain. Hence R is prime .Suppose $x^2 = 0$, then we have x =0 .
As R is a domain. Therefore, xRx =0 . And R is QSC ring .
Remark: The **Theorem (2.34)** and **Theorem (2.35)** are analogs of **Theorem (2.18)** .
Corollary 2.36: An ideal I of R is prime(semiprime) and R/I is QSC ring if and only if I is a completely prime (completely semiprime) .
Corollary 2.37: Let R be a semiprime ring. Then the following are equivalent :
1- R is QSC ring .
2- 2- R is reduced .
3- R is SC ring .
Proof : By Theorem (2.34), we get (1) which is equivalent to (2)
and (2) is equivalent to (3) by Theorem (2.18).
Example 2.38: There exists SC rings which are not HSC rings . **[11]**
Proof : Let R be the localization of the ring ℤ at the prime (3)
Let Q be the ring of quaternions over R, that is, the basis 1 , i ,
j , k and multiplication satisfying $i^2 = j^2 = k^2 = -1$, ij = k = -ji . Then Q is non-commutative domain, so it is SC ring.
However, J(Q) =3Q and Q/J(Q) is isomorphic to the 2×2 full matrix ring over $\mathbb{Z}_3$. So Q/J(Q) is not SC ring as it is not abelian . Thus Q is not HSC rings .
Definition 2.39: A ring R is called homomorphically Q- semicommutative ring(simply HQSC) if R/I is Q-semicommutative ring for every proper ideal I in R .
Note 2.40: It is clear that every HSC rings are HQSC rings.
Note 2.41: We see that every HQSC ring is QSC, but the converse is not true in general, for Example :-
Example 2.42: Let S be a ring in the **Example(2.38)** , then S/J(S) is isomorphic to the 2- by-2 full matrix over Z_3. S is QSC as it is domain .
We have $\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}^2 = 0$. But $\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix} = \begin{bmatrix}1 & 2\\1 & 2\end{bmatrix} \neq 0$
Hence S/J(S) is not QSC ring .Therefore, S is not HQSC ring .
Let's recall the definition of 2-primal .
A ring R is called 2-primal if P(R) = N(R) ,where P(R) used to denote the prime radical and N(R) the set of all nilpotent element is R .
Theorem 2.43: If ring R is HQSC then it is 2-primal .
Proof : Let I be a prime ideal, R/I is QSC as R is HQSC. Then I is completely prime by **corollary (2.36)** . Thus R is 2-primal by using Shin's result [5].
We get an immediate consequence of Theorem(2.34) in the following Corollary.
Corollary 2.44: Let R be a ring . The following are equivalent:
1- R is regular, prime and QSC.
2- R is a division ring .
Proof :⟹ Let R be a prime and QSC .Then R is a domain by Theorem (2.23). A regular domain is a division ring by [17].
Conversely : Assume that R is a division ring then it is a domain, so it is
Prime and QSC ring by theorem(2.35) . it is clear that every division ring is regular .
Theorem 2.45: Let R be a regular ring then the following are equivalent:
1- R is abelian .
2- R is HQSC ring .
3- R/P is a division ring for any prime ideal P of R .
4- R is 2-primal .
5- R is left (right) due .
Proof : (1)⟹(2) .Let R be an abelian ring . For every proper ideal I of R.
Let $x^2 \in I$, since R is regular, there exists some a ∈R such that x = xax ,
so xa is an idempotent of R .Hence, it is central . So $x = xax = x^2a$. Thus
x ∈I as $x^2 \in I$ .Hence xRx ⊆I .So R/I is QSC ring and R is HQSC ring .
(2)⟹(1) . By assumption it is QSC ring. If $x^2 = 0$, then xRx=0 . As R is regular, x ∈ xRx and so x =0 Therefore, R is reduced. By theorem (2.29), we get R is abelian.
(2)⟹(3) . Let P be a prime ideal of R, where R is HQSC ring .Then R/P is QSC ring , prime and regular .Hence, R/P is a division ring by Corollary (2.44)
(3)⟹(4) . Assume that R/P is division ring , for all prime ideal P of R .
Then R/P is reduced. Hence, P/P(R) is reduced , where P(R) is prime radical of R .Therefore, R is 2-primal by Lemma (2.24) .
(4)⟹(1) and (1)→(5) are clear by **Theorem(2.29)** .
Note that (1)→(3) clear by the result in [18]
Theorem 2.46: If I is a prime(semiprime) ideal of R, and R/I is SC ring, then I is completely prime (completely semiprime) ideal . **[13]**
We get another version of Theorem(2.18) in the similar proof .
Proof : Assume that R/I is an SC ring .If I is a prime (semiprime) ideal of R, then R/I is a prime(semiprime) ring. So R/I is a domain (reduced) ring by Theorem(2.36). Hence, I is completely prime(completely semiprime) ideal. Thus Theorem (2.18) can be as corollary of Theorem(2.46) when I={0}.
Definition 2.47 An ideal I of a ring R is called 2-primal if P(R/I) = N(R/I) . **[19]**

Definition 2.48: A ring is called strongly 2-primal if every proper ideal I of R is 2-primal, where the term proper means only I ≠R . **[13]**

Note 2.49: We see that a ring R is 2-primal if and only if the zero ideal is 2-primal and hence that every strongly 2-primal ring is 2-primal. But, the converse is not true by Example (2.7) in [20] .

The following Theorem was proved by Shin [2] .

Theorem 2. 50 : A ring R is strongly 2-primal if and only if every prime ideal I of R is completely prime .**[2]**

Theorem 2.51: A ring R is NI if and only if every minimal strongly prime ideal of R is completely Prime .**[21]**

Corollary 2.52[21]

If R/P is SC ring for any minimal strongly prime ideal P of a ring R, then R is NI .

proof: By Theorem (2.46) and by Theorem (2.51) we get the proof .

Then we have the following result which is analog of Corollary (2.44) .

Theorem 2.53

Let R be a strongly 2-primal if and only if R/P is QSC ring(SC) for all prime ideal P in R .

proof : Assume P be a prime ideal of R .

R/P is QSC ring(semicommutative) if and only if R/P is

domain by Theorem (2.30) or by Corollary (2.33) .

That is , P is completely prime and R is strongly 2-primal if and only if

every prime ideal of R is completely prime by [2].So the result follows.

Remark 2.54

By Theorem (2.53) we easily get every HQSC ring(HSC) which is strongly 2-primal .

Corollary 2.55

Let R/P be a QSC for all prime ideals in R, then R/P(R) is a strongly 2-primal ring .

proof : R is a strongly 2-primal ring which is equivalent to R/P(R) and it is a strongly 2-primal ring by [20] and by Theorem(2.52), we get the following result .Then the following result in [13] can be viewed as a corollary of the preceding Theorem .

Corollary 2.56

Let R be a Von Neumann regular ring then the following are equivalent :

1- R is 2-primal .
2- R is strongly 2-primal .

Proof : For (2)⟹(1) is clear .

For (1)⟹(2) . Let R be a von Neumann regular 2-primal ring , then R is abelian Theorem(2.28). Therefore, R is HQSC ring by Theorem (2.44) in particular R/P is QSC ring for all prime ideals P of R by Theorem(2.53) we get R is strongly 2-primal .

Note 2.57

It is important to combine Theorem (2.29,Corollary (2.36,Theorem (2.45) and Corollary(2.52) we get the following Theorem:

Theorem 2.58: Let R is regular Von Neumann then the following are equivalent :

1- R is abelian .
2- R is right(left) duo .
3- R is reduced .
4- R is revisable .
5- R is SC .
6- R is 2-primal .
7- R is NI .
8- R is QSC .
9- R is HSC .
10- R is HQSC .
11- R/P is a division ring for any prime ideal P of R .
12- R is strongly 2-primal .

Definition 2.59[9]: Let R be a ring an element x in R is called entire if it is not a zero divisor .

Definition 2.60[9]: A commutative ring R and S is multiplicatively closed subset of R with 1∈S and 0∉S .We define $S^{-1}R$ to be the set of all pair (r,s), when r∈ R, s∈ S modulo the equivalence ~ where $(r_1,s_1)\sim(r_2,s_2) \Leftrightarrow r_1s_2s = r_2s_1s$ for some s∈ S. $S^{-1}R$ is called the localization of R at S .

Theorem 2.61: Let R be a ring, and let D be a multiplicatively closed subset of R consisting of a central entire elements where $D^{-1}R = \{d^{-1}a \mid d\in D, a\in R\}$. Then the following are equivalent:

1- R is QSC .
2- $D^{-1}R$ is QSC .

Proof : (1)⟹(2) . Let $\beta^2=0$ with $\beta = y^{-1}x$, y∈ D and x∈ R. Then

$y^{-1}xy^{-1}x=0$. So $x^2 y^{-1} y^{-1} = 0$ as y^{-1} is central .

Hence $x^2 = 0$ .For any $d^{-1}s \in D^{-1}R$, where s ∈R , d∈ D .

We have xsx = 0 as R is QSC .

Hence $y^{-1}xd^{-1}sy^{-1}x = xsxy^{-1}d^{-1}y^{-1} = 0$. So $D^{-1}R$ is QSC .

(2)⟹(1) . It is clear since the class of QSC ring is closed under subring .

Definition 2.62[22] The ring of Laurent polynomial in x, coefficients in a ring R, consists of all formal sums $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where m_i ∈ R and k, n are (possibly negative) integer. We denote this ring by $R[x; x^{-1}]$.

Corollary 2.63

Let R be a ring, R[x] is QSC if and only if the ring of Laurent polynomials

$R[x,x^{-1}]$ is QSC .

Proof : To prove the necessity as R[x] is a subring of $R[x,x^{-1}]$.

Let D = { 1 , x , x^2 , …} . Then D is a multiplicatively closed under subset of R[x] consisting of central entire element and $R[x , x^{-1}] =$

$D^{-1}R[x]$. Hence $R[x , x^{-1}]$ is QSC by Theorem (2.61) .

Theorem 2.64 : Let R be a ring and e be a central idempotent . Then the following are equivalent:

1- R is QSC ring .
2- eRe = eR and (1-e)R(1-e) = (1-e)R are QSC ring.

proof : (1)⟹(2) : It is clear since eR and (1-e)R are subring of R . (2)⟹(1) : For any x∈ R, let $x^2 = 0$.

Then $(ex)^2 = exex = ex^2 = 0$. So $exeRex = 0$ and $exRx = 0$.
Similarly: We have $((1-e)x)^2 = (1-e)x^2 = 0$ so $(1-e)x(1-e)R(1-e)x = 0$.
Which implies $(1-e)xRx = 0$.
Hence $xRx = exRx + (1-e)xRx = 0$. R is QSC ring .
($R=eR\oplus(1-e)R$, and QSC is preserved under direct sum) .

Example 2.64

Let $R=T_4(\mathbb{Z})$ then $\begin{bmatrix}0&1&0&1\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}1&0&0&1\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$

But $\begin{bmatrix}0&1&0&1\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}0&1&0&1\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix} = \begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \neq 0$.

Thus R is not QSC ring .

Let $e = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$. Then $eRe = \begin{bmatrix}\mathbb{Z}&\mathbb{Z}&0&0\\\mathbb{Z}&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$ and $(1-e)R(1-e) = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&\mathbb{Z}&\mathbb{Z}\\0&0&0&\mathbb{Z}\end{bmatrix}$ are QSC ring .

So the condition that e is central is necessary in Theorem (2.63) .

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حول تعميم على الحلقات شبة الابدالية

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الملخص

نحن في هذه البحث قدمنا مفهوم الحلقات الشبه الابدالية من نوع Q التي هي عبارة عن تعميم للحلقات الشبه الابدالية. في سنة 1973 قدم الباحث في مجال الرياضيات (Shin) مفهوم الحلقات الشبه الابدالية والتي أجرى عليها العديد من التعميمات حيث ان كثير من الباحثين اجروا وقدموا دراسات على هذا المفهوم. أيضا في هذه الرسالة تمت مناقشة خصائص مختلفة على هذا المفهوم (الحلقات الشبة الابدالية نوع Q) ولاسيما الخصائص والشروط التي تبدوا قوية ومنها (NI , 2-Primal) لقد تم التركيز بشكل خاص في هذه الرسالة على اوجه التشابه والاختلاف بين كل من الحلقات الشبه الابدالية والحلقات الشبه الابدالية من نوع Q. نحن ايضا استطعنا في هذه الرسالة ان نبين ان كل حلقه شبه ابدالية من نوع Q هي حلقة شبه ابدالية واستطعنا ان نقدم بعض الامثلة التي تبين العكس. وايضا بينا ان حلقة القسمة R/I هي حلقة شبة ابدالية من نوع Q.