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# Dynamical behavior of the Family of cubic functions

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**ABSTRACT**

May R.M. gave the example of the family of cubic maps of the interval $[-1,1]$. Rogers T. D. extends the analysis of May beyond that region .
In this paper we are trying to introduce comprehensive study of the cubic family which defined in the form:

$f_\alpha(x)=\alpha x^3+(1-\alpha)x \qquad (1)$

The fixed points of the family are determined and described according to the values of the parameter $\alpha$ . Dynamical and chaotic behaviours of the family discussed according to different definitions of chaos and via conjugacy .

## 1. Introduction

The word chaos in modern dictionaries, is defined as "total disorder and confusion". **[3]** Edward Lorenz simplified chaos in short words "when the present determines the future , but the approximate present does not approximately determine the future" . Chaos as an property is observed after a period of time which implies a system which could possibly display non-chaos as usual in the initial stages of iteration though highly and easily chaotic after a few iteration .

## 2.Fixed points.

A point whose iterates are the same point is called a fixed point . In other words it's not changed under the effect of the function motion , therefore fixed points very important in the study of the dynamics of functions.

**Definition. 2.1 [5],[1]**

Let $p$ be a point in the domain of the function $f$ . Then $p$ is called a fixed point of $f$ if $f(p)=p$. Graphically , the point $p$ is a fixed point of the function $f$ if and only if the graph of $f$ touches (or crosses) the line $y=x$ at $(p,p)$ .

**Definition. 2.2 [5]**

Let $p$ be a fixed point of the function $f$. The point $p$ is called attracting fixed point if there exists an interval $(p-\varepsilon,p+\varepsilon)$ such that if $x$ is in the domain of $f$ and $x\in(p-\varepsilon,p+\varepsilon)$ then $f^{[n]}(x)\to p$ as $n$ increases without bound , such a point also called asymptotically stable. i.e. attracting fixed point attracts the iterates of the near points to itself in some interval.

**Definition. 2.3 [5]**

Let $p$ be a fixed point of the function $f$. The point $p$ is called repelling fixed point if there exist an interval $(p-\varepsilon,p+\varepsilon)$ such that if $x$ is in the domain of $f$ , $x\in(p-\varepsilon,p+\varepsilon)$ and $x\neq p$ then $|f(x)-p|>|x-p|$ . i.e. it repeals the iterations of near points to some distance.

## 3.Dynamical behaviour of the family

**Theorem 3.1 [5]**

Suppose that $f$ is differentiable at a fixed point $p$.

i. If $|f'(p)|<1$ , then the point $p$ is called attracting fixed point.

ii. If $|f'(p)|>1$ , then the point $p$ is called repelling fixed point.

iii. If $|f'(p)|=1$ , then $p$ can be attracting , repelling , or neither.

To find fixed points of any function $f$ we just solve the equation $f(x)=x$.

So , solving the equation

$f_\alpha(x)=x$

$\alpha x^3+(1-\alpha)x=x$

We obtain three fixed points $x_1=0$ , $x_2=1$ and $x_3=-1$

Now we investigate each of them in three cases.

**Case1.** $x=0$

$f_\alpha(x)=\alpha x^3+(1-\alpha)x$

$f'_\alpha(x)=3\alpha x^2+(1-\alpha)$

$\Rightarrow f'_\alpha(0)=3\alpha(0)+(1-\alpha)$

$\Rightarrow f'_\alpha(0)=1-\alpha$

$\Rightarrow |f'_\alpha(0)|=|1-\alpha|$

Then if $\alpha>2$ or $\alpha<0$ that leads to make $|f'_\alpha(0)|>1$ which means that a fixed point $x=0$ is a repelling fixed point by theorem (3.1).

If $0<\alpha<2$ that leads to make $|f'_\alpha(0)|<1$ which means that a fixed point $x=0$ is an attracting fixed point by theorem (3.1).

If $\alpha=2$ that leads to make $|f'_\alpha(0)|=1$ and the theorem above doesn't tell us anything about the fixed point, therefore we must use another criterion using Schwarzian derivative which defined as follows :**[3]**

$$Sf(x)=\frac{f'''(x)}{f'(x)}-\frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2$$

Since $x=0$ and $\alpha=2$

$\Rightarrow f'_\alpha(x)=3\alpha x^2+(1-\alpha)$

$\Rightarrow f''_\alpha(x)=6\alpha x$

$\Rightarrow f'''_\alpha(x)=6\alpha$

$$Sf(0)=-f'''(0)-\frac{3}{2}[f''(0)]^2$$

$$=-6\alpha-\frac{3}{2}[6\alpha(0)]^2$$

$=-12$

Since , $Sf(0)<0$ then a fixed point $x=0$ is attracting when $\alpha=2$ and it's basin of attracting is $B_0=(-1,1)$ .

**Case2 .** $x=1$

$f'_\alpha(1)=2\alpha+1$

$|f'_\alpha(1)|=|2\alpha+1|$

so, if $\alpha>0$ or $\alpha<-1$ $\Rightarrow$ $|f'_\alpha(1)|>1$ which makes the fixed point $x=1$ is a repelling fixed point.

But, if $-1<\alpha<0$ then $|f'_\alpha(1)|<1$ which makes the fixed point $x=1$ is an attracting fixed point.

If $\alpha=-1$ then $|f'_\alpha(1)|=1$ and since $f'_\alpha(1)=-1$ we need to use Swarzian derivative again to determine the character of the point.

Calculating the value of the derivative we obtain:

$Sf(1)=-48$

Since $Sf<0$ then , the fixed point $x=1$ is attracting and the basin of attraction $B_0=(0,1]$ .

**Case3.** $x=-1$

We obtain that $|f'_\alpha(-1)|=|2\alpha+1|$ , therefore the same discussion repeated as in case 2 .

so, if $\alpha>0$ or $\alpha<-1$ $\Rightarrow$ $|f'_\alpha(1)|>1$ which makes the fixed point $x=-1$ is a repelling fixed point.

But, if $-1<\alpha<0$ then $|f'_\alpha(1)|<1$ which makes the fixed point $x=-1$ is an attracting fixed point.

If $\alpha=-1$ then $|f'_\alpha(1)|=1$ and since $f'_\alpha(1)=-1$ we need to use Swarzian derivative again to determine the character of the point.

Calculating the value of the derivative we obtain:

$Sf(1)=-48$

Since $Sf<0$ then , the fixed point $x=-1$ is attracting and the basin of attracting $B_0=[-1,0)$ .

**4. Chaotic behaviour of the family**

**Definition . 4.1 [3],[5],[2]**

Let $J$ be a bounded interval, and $f:J\to J$ continuously differentiable on $J$, then $\lambda(x)$ which defined as follows:

$\lambda(x)=\lim_{x\to\infty}\frac{1}{n}\ln|(f^{[n]})'(x)|$ , is called Lyapunov exponent of $f$ .

**Definition . 4.2 [5]**

Let $J$ be an interval , and suppose that $f:J\to J$ . Then $f$ has sensitive dependence on initial conditions at $X$ if , there exist $\varepsilon>0$ such that for each $\delta>0$ , there is $y\in J$ and $n\in\mathbb{Z}^+$ such that $|x-y|<\delta$ *and* $|f^{[n]}(x)-f^{[n]}(y)|>\varepsilon$

**Definition . 4.3**

We will use the next formula to calculate Lyapunov exponent **[5]**

$\lambda(x)=\lim_{x\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\ln|f'_\alpha(x)|$ then we obtain

$$\lambda(x)=\lim_{x\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\ln|3x^2+(1-\alpha)|$$

$\Rightarrow$ $\lambda(x)>0$

**Definition . 4.4 [5]**

A function $f$ is said to be chaotic if it is satisfied at least one of the following conditions:

i. $f$ has positive Lyapunov exponent at each point in it's domain.

ii. $f$ has sensitive dependence on initial conditions on it's domain.

**Then , the cubic family is chaotic by Guilick definition of chaos since it has positive Lyapuniv exponent .**

**5. Conjugacy**

In some cases , it's difficult to show if one function has features as transitivity or existence of a dense set of periodic points and it's easer to find other function which has these features and conjugate to our function , specially existence of period - 3 points for the cubic family will give an important clue about the chaotic behaviour of the family, but to find out such points is not as easy **[10]**. If two functions are conjugate to one another, then one function inherits such properties as transitivity and the existence of a dense set of periodic points from the other one.

**Definition. 5.1 [2],[5],[7]**

Let $J$ and $K$ be intervals , and suppose that $f:J\to J$ and $g:K\to K$. then $f$ and $g$ are conjugate if there exist a homeomorphism $h:J\to K$ such that $h\circ f=g\circ h$ and in this case we write $f\underset{h}{\approx}g$ .

**Theorem. 5.2** Suppose $f:[a,b]\to\mathbb{R}$ is a continuous map. If $f$ has an orbit of period - 3, then $f$ is chaotic. **[7]**

**Theorem. 5.3** If $f:X\to X$ *and* $g:Y\to Y$ are conjugate maps via conjugacy $h$ , then $f$ is chaotic iff $g$ is chaotic. **[4]**

Let $g$ be defined as follows:

$$g(x)=\begin{cases}3x & , \quad 0\le x<\frac{1}{3}\\ 2-3x & , \quad \frac{1}{3}\le x<\frac{2}{3}\\ 3x-2 & , \quad \frac{2}{3}\le x\le 1\end{cases}$$

We'll show that $f\underset{h}{\approx}g$ in case if $\alpha=4$ , i.e. $f(x)=4x^3+3x$ and $h(x)=\cos(\pi x)$.

Since $f\circ h(x)=\cos(3\pi x)$

And $h\circ g(x)=\cos(3\pi x)$

Then $f\circ h(x)=h\circ g(x)$

To show that $f$ has perid-3 point

let $x=\frac{2}{7}$ , then $\left\{\frac{2}{7},\frac{6}{7},\frac{4}{7}\right\}$ a 3-cycle for $g$ .

Then $g$ has a periodic orbit **[5]** and therefore it is chaotic by theorem 5.2 .

So the cubic function $f$ is chaotic by theorem 5.3

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# السلوك الديناميكي لعائلة الدوال التكعيبية


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**الملخص**

لقد قدم May R.M. مثالا لعائلة الدوال التكعيبية على الفترة $[-1,1]$ ، وقام Rogers T. D. بتوسيع تحليل ماي على فترة أكبر. نقدم في هذه الورقة دراسة شاملة لعائلة الدوال التكعيبية المعرفة بالشكل: $f_\alpha(x)=\alpha x^3+(1-\alpha)x$ . حيث تم ايجاد النقاط الثابتة للعائلة وتم وصفها لقيم مختلفة للمعلمة $\alpha$ . كما تم مناقشة السلوك الديناميكي والفوضوي للعائلة نسبة الى مختلف تعاريف الفوضى ومن خلال الدالة المرافقة.