Tikrit Journal of Pure Science 24 (1) 2019
ISSN: 1813 – 1662 (Print)
E-ISSN: 2415 – 1726 (On Line)

TJPS

Tikrit Journal of Pure Science

Journal Homepage: http://tjps.tu.edu.iq/index.php/j

The Othogonality of Martingale in Birkhoff's sense and others

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https://doi.org/10.25130/tjps.v24i1.340

ARTICLE INFO.

Article history:
-Received: 3 / 9 / 2018
-Accepted: 24 / 10 / 2018
-Available online: / / 2018

Keywords: Orthogonality of martingale, Birkhoff 's orthogonality , Roberts's orthogonality, Singer's orthogonality, Carlsson's orthogonality.

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ABSTRACT
Orthogonality is one of an important the concepts in Mathematics , therefor it will be discussed in this paper, the orthogonality of martingale according to Birkhoff 's, Roberts's, Singer's, Carlsson's sense for orthogonality and the conditions that are needed to have orthogonality.

1. Introduction

Let $\{X_1, X_2 \ldots\}$ be a sequence of integrable random variable on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ an increasing sequence of sub $\sigma-$ field of $\mathcal{F}$, X_n is $\mathcal{F}_n$ - measurable that is $X_n : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. $\{X_n, \mathcal{F}_n\}$ is said to be martingale iff $\forall n, E[X_{n+1} | \mathcal{F}_n] = X_n$ a.e.[1]

If $E(|Y|^P) < \infty$ It's said Y $\in L^P(\Omega, \mathcal{F}, P)$ for $P \in [1, \infty)$ and the norm defined as :

$$\|Y\|_P = \left\{E(|Y|^P)\right\}^{\frac{1}{P}}.[2]$$

James was the first one Who studied Birkhoff 's properties of orthogonality, therefor this orthogonality is called Birkhoff- James.[3]

Ash proved that the martingale difference is orthogonal in a Hilbert space $L^2(\Omega, \mathcal{F}, P)$ by usual orthogonality[1].

In 1934 Roberts introduced his orthogonality as Roberts's orthogonality and in 1935 Birkhoff introduced his orthogonality as Birkhoff 's orthogonality , which was one of the most important orthogonality senses in normed space . [4]

Singer's orthogonality was introduced by singer in 1957. Therefor, the orthogonality of martingale will be discussed in normed space according to these senses.[5]

A functional is a mapping g of an element in a normed space $L^P(\Omega, \mathcal{F}, P)$ to an element in its scalar field .

A functional is called linear when it satisfies,

$g(aX + bY) = ag(X) + bg(Y)$

for every scalars and $X, Y \in L^P(\Omega, \mathcal{F}, P)$.[6]

We say that functional g is bounded functional if there exist scalar $k \geq 0$ such that

$|g(X)| \leq K \|X\|_P$

for every $X \in L^P(\Omega, \mathcal{F}, P)$

The space of all bounded linear functionals on $L^P(\Omega, \mathcal{F}, P)$ is called the dual space of $L^P(\Omega, \mathcal{F}, P)$ and denoted by L^* .[6]

2- Main results

Definition(2.1)[7]

In a vector normed space $L^P(\Omega, \mathcal{F}, P)$, Z is called Birkhoff – James orthogonal to W and denoted by $Z \perp_B W$ if

$\|Z + aW\|_P \geq \|Z\|_P$

for any real number a .

Theorem (2.2)[7]

Let X and Y belongs to a normed space $L^P(\Omega, F, P)$, then $X \perp_B Y$ if and only if there exist $g \in L^* \setminus \{0\}$ such that $|g(X)| = \|g\|_P \|X\|_P$.

Lemma(2.3)

Let $E : L^P(\Omega, F, P) \to \bar{R}$ be a conditional expectation, then E is bounded linear operator.

Proof:

Since $E(aX + bY \mid F) = aE(X \mid F) + bE(Y \mid F)$ then E is linear operator and $\|E(X \mid F)\|_P \le \|X\|_P$ that is E is bounded by $K = 1$.

Theorem(2.4)

Let $X_n \in L^P(\Omega, F, P)$ and (X_n, F_n) be a zero – mean martingale then $X_n \perp_B X_{n+1}$.

Proof:

If $g(X_n) = E(X_n \mid F_n)$, we define

$|g(X_n)| = \|g(X_n)\|_P$

$|g(X_n)| = \|E(X_n \mid F_n)\|_P$

$= \|X_n E(1 \mid F_n)\|_P$ (since X_n is F_n - measurable)

$= |E(1 \mid F_n)| \|X_n\|_P$ (by property of norm)

$= \|E(1 \mid F_n)\|_P \|X_n\|_P$

$= \|E\|_P \|X_n\|_P$

$= \|g\|_P \|X_n\|_P$.

Since $|g(X_n)| = \|g\|_P \|X_n\|_P$ and $E(X_{n+1} \mid F_n) = 0$, then(by (2.2)) $\sum_{i=1}^{m} \alpha_i \|\beta_i X + \gamma_i Y\|_P^2 = 0$

$X_n \perp_B X_{n+1}$

Definition(2.5)[8]

In a normed space $L^P(\Omega, F, P)$, Z is said to be singer orthogonal to W and denoted by $Z \perp_S W$ if either $\|Z\|_P \|W\|_P = 0$ or $\|Z + W\|_P = \|Z - W\|_P$.

Theorem(2.6)

If $E(|X_n|^P) < \infty$ for all n , X_n is a martingale and independent if $E(X_{n+1}^P \mid F_n) = 0 \; \forall P \in [1, \infty)$, then $X_n \perp_S X_{n+1}$.

Proof:

$\|X_n\|_P^P \|X_{n+1}\|_P^P = E(|X_n|^P) E(|X_{n+1}|^P)$

$= E[X_n^P X_{n+1}^P]$ (by independence)

$= E[E(X_n^P X_{n+1}^P \mid F_n)]$

$= E[X_n^P E(X_{n+1}^P \mid F_n)]$

(Since X_n^P is F_n - measurable).

$= 0$

Therefor $\|X_n\|_P \|X_{n+1}\|_P = 0$

Hence $X_n \perp_S X_{n+1}$.

Definition(2.7)[4]

In a normed space $L^P(\Omega, F, P)$, Z is said to be Roberts orthogonal to W and denoted by $Z \perp_R W$ if the equality

$\|Z + \alpha W\|_P = \|Z - \alpha W\|_P$.

holds for any real number α .

Theorem(2.8)

If X_n belong to a Hlibert space $L^2(\Omega, F, P)$ and (X_n, F_n) is a martingale then martingale differences are orthogonale in the sense of Roberts.

Proof:

$\|(X_m - X_{m-1}) + \alpha(X_n - X_{n-1})\|_2^2$

$= E[[(X_m - X_{m-1}) + \alpha(X_n - X_{n-1})]^2]$

$= E[(X_m - X_{m-1})^2 + 2\alpha(X_m - X_{m-1})(X_n - X_{n-1}) + \alpha^2(X_n - X_{n-1})^2]$

$= E(X_m - X_{m-1})^2 + 2\alpha E[(X_m - X_{m-1})(X_n - X_{n-1})] + \alpha^2 E(X_n - X_{n-1})^2$

$= E(X_m - X_{m-1})^2 + 2\alpha E[E[(X_m - X_{m-1})(X_n - X_{n-1})] \mid F_n] + \alpha^2 E(X_n - X_{n-1})^2$

$= E(X_m - X_{m-1})^2 + 2\alpha E[(X_m - X_{m-1}) E[(X_n - X_{n-1})] \mid F_n] + \alpha^2 E(X_n - X_{n-1})^2$

Since $X_m - X_{m-1}$ is F_n -measurable.

$E[(X_n - X_{n-1})] \mid F_n] = X_m - X_m = 0$

(by property of martingale)

$= E(X_m - X_{m-1})^2 + \alpha^2 E(X_n - X_{n-1})^2$

$= \|X_m - X_{m-1}\|_2^2 + \alpha^2 \|X_n - X_{n-1}\|_2^2$ …(1)

Similarity

$\|(X_m - X_{m-1}) - \alpha(X_n - X_{n-1})\|_2^2$

$= \|X_m - X_{m-1}\|_2^2 + \alpha^2 \|X_n - X_{n-1}\|_2^2$ …(2)

From (1) and (2) we have,

$\|(X_m - X_{m-1}) + \alpha(X_n - X_{n-1})\|_2$

$= \|(X_m - X_{m-1}) - \alpha(X_n - X_{n-1})\|_2$

$(X_m - X_{m-1}) \perp_R (X_n - X_{n-1})$.

Definition(2.9)[9]

Let X , Y belong to a normed space $L^P(\Omega, F, P)$, and m be a positive integer . Then X is said to be orthogonal in the sense of Carlsson to Y and denoted by

$X \perp_C Y$ if and only if $\sum_{i=1}^{m} \alpha_i \|\beta_i X + \gamma_i Y\|_P^2 = 0$.

Where $\alpha_i, \beta_i, \gamma_i$ are real number such that

$\sum_{i=1}^{m} \alpha_i \beta_i^2 = \sum_{i=1}^{m} \alpha_i \gamma_i^2 = 0$, and $\sum_{i=1}^{m} \alpha_i \beta_i \gamma_i = 1$.

Theorem(2.10)

Let $E(|X_n|^P) < \infty$ and X_n be a martingale such that $E(X_{n+1}^{P-K} \mid F_n) = 0$, $\forall P \in [1, \infty)$ and $K = 0, 1, \dots, P$ then $X_{n+1} \perp_C X_n$.

Proof:

$\sum_{i=1}^{m} \alpha_i \|\beta_i X_{n+1} + \gamma_i X_n\|_P^2 = \sum_{i=1}^{m} \alpha_i \left\{ \left\{ E(\beta_i X_{n+1} + \gamma_i X_n)^P \right\}^{\frac{1}{P}} \right\}^2$

$= \sum_{i=1}^{m} \alpha_i \left\{ E[\sum_{K=0}^{P} \binom{P}{K} (\beta_i X_{n+1})^{P-K} (\gamma_i X_n)^K] \right\}^{\frac{2}{P}}$

(by binomial theorem)

$$=\sum_{i=1}^{m}\alpha_i\left\{\sum_{K=0}^{P}\binom{P}{K}\beta_i^{P-K}\gamma_i^{K}E[X_{n+1}^{P-K}X_n^{k}]\right\}^{\frac{2}{P}}$$

$$=\sum_{i=1}^{m}\alpha_i\left\{\sum_{K=0}^{P}\binom{P}{K}\beta_i^{P-K}\gamma_i^{K}E(E[X_{n+1}^{P-K}X_n^{k}\mid \mathrm{F}_n])\right\}^{\frac{2}{P}}$$

$$=\sum_{i=1}^{m}\alpha_i\left\{\sum_{K=0}^{P}\binom{P}{K}\beta_i^{P-K}\gamma_i^{K}E(X_n^{K}E[X_{n+1}^{P-K}\mid \mathrm{F}_n])\right\}^{\frac{2}{P}}$$

$$=0$$

Since X_n^K is F_n - measurable.

Hence $X_{n+1}\perp_C X_n$.

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الملخص

التعامد من المفاهيم المهمة في الرياضيات، لذلك نوقش في هذا البحث تعامد المارتنيكل حسب مفهوم بيرخوف, روبرتس, سنكر, وكارلسون للتعامد وماهي الشروط الواجب توفرها للحصول على التعامد.