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ON (sub- super) asymptotic martingales

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ABSTRACT

In this paper we intoduce a new class of definitions (sub - super) asymptotic martingale through the concept of asymptotic martingale. we investigate and prove some properties of asymptotic martingale and (sub - super) asymptotic martingale .

1- Introduction

Theory of asymptotic martingales(amart) has been developed and extentively studied in recent years by Bellow[5], Edgar and Sucheston [7], Chacon and Sucheston[6]. These authors are the first to believe that the notion merits a name: asymptotic martingale. A systematic presentation of amart theory paralleling the martingale theory, including for the first time the optional sampling theorem, the Riesez decomposition ,the descending and the parameter cases ,was given by edgar.

among other. It was show that every real valued amart and every vector- valued uniform amart has aRies decomposition.

the amart combines several useful properties of the martingale, submartingale , supermartingale. Thus the class of martingales is closed under linear combinations ,the class of supermartingales under infimum, the class of submartingales under supremum , but the class of amart is closed under all three operations[2].

In (1975) R .V .chacon and L. Sucheston, reduced on convergence of vector asymptotic martingale [8].

.we recall that the definition of asymptotic martingale, A sequence $\{X_n, \mathcal{F}_n, n\geq 1\}$ **if and only if** for every $\varepsilon > 0$, there exsit $t_0 \in T$ such that for every $t,\sigma \in T$, $t,\sigma \geq t_0$ a.s. We have

$\|\int X_t \, dp - \int X_\sigma \, dp\| < \varepsilon$.[3] .

our aim in this paper is to study and proves some properties of the asymptotic martingale and (sub,super) asymptotic martingale and every martingale is asymptotic martingale .

2 – (sub - super)asymptotic martingale.

In this section we introduce a new class of definitions(sub - super)asymptotic martingale and prove some properties of asymptotic martingale and (sub - super) asymptotic martingale .

Definition 2.1

A sequence $\{X_n, \mathcal{F}_n, n\geq 1\}$is called a sub asymptotic martingale (for short - subamart) **if and only if** for every $\varepsilon > 0$, there exsit $t_0 \in T$ such that for every $t, \sigma \in T$, $t,\sigma \geq t_0$ a.s.

We have

$\|\int X_\sigma \, dp - \int X_t \, dp\| \leq \varepsilon$.

Definition 2.2

Asequence $\{X_n, \mathcal{F}_n, n\geq 1\}$is called a super asymptotic martingale (for short - superamart) **if and only if** for every $\varepsilon > 0$, there exsit $t_0 \in T$ such that for every $t, \sigma \in T$, $t,\sigma \geq t_0$ a.s.

We have

$\|\int X_t \, dp - \int X_\sigma \, dp\| \leq \varepsilon$.

Remark 2 .3

1-Martingale $\Rightarrow$ asymptotic martingale $\Rightarrow$ super asymptotic martingale.

2- sub asymptotic martingale $\nLeftrightarrow$ super asymptotic martingale.

Example 2.4

Let X_n be integrable random variable and $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots$ be a filter on probablitiy space (Ω, F, P), $E|X| < \infty$ and define $X_n = a/2$, then $\{X_n, \mathcal{F}_n\}$ is asymptotic martingale.

Theorem 2.5

If $\{X_n, \mathcal{F}_n, n\geq 1\}$ and $\{Y_n, \mathcal{F}_n, n\geq 1\}$are asymptotic martingale,then $\{X_n + Y_n, \mathcal{F}_n, n\geq 1\}$is asymptotic martingale

With corresponding statements for subasymptotic martingale and superasymptotic martingale.

Proof

Since $\{X_n, \mathcal{F}_n, n\geq 1\}$is asymptotic martingale , for every $\varepsilon > 0$, there exsit $t_0 \in T$ such that for every $t, \sigma \in T$, $t,\sigma \geq t_0$,

Then $\|\int X_t\, dp - \int X_\sigma\, dp\| < 1\backslash 2\varepsilon \ldots (2.5.1)$

since $\{Y_n, \mathcal{F}_n, n\geq 1\}$is asymptotic martingale, for every $\varepsilon > 0$, there exsit $t_0 \in T$ such that for every $t, \sigma \in T$, $t,\sigma \geq t_0$,

Then $\|\int Y_t\, dp - \int Y_\sigma\, dp\| < 1\backslash 2\varepsilon \ldots\ldots (2.5.2)$

We want to prove that

$\|\int (X_t + Y_t)dp - \int (X_\sigma + Y_\sigma)dp\| < \varepsilon$

We have from (2.5.1) and (2.5.2) we get

$\|\int (X_t + Y_t)dp - \int (X_\sigma + Y_\sigma)dp\| = \|\int X_t\, dp - \int X_\sigma\, dp + \int Y_t\, dp - \int Y_\sigma\, dp\| \leq \|\int X_t\, dp - \int X_\sigma\, dp\| + \|\int Y_t\, dp - \int Y_\sigma\, dp\| < 1\backslash 2\varepsilon + < 1\backslash 2\varepsilon = \varepsilon$

Thus$\{X_n + Y_n, \mathcal{F}_n, n\geq 1\}$is asymptotic martingale .

Theorem 2.6

If $\{X_n, \mathcal{F}_n, n\geq 1\}$ and $\{Y_n, \mathcal{F}_n, n\geq 1\}$are asymptotic martingale, then $\{X_n - Y_n, \mathcal{F}_n, n\geq 1\}$is asymptotic martingale

With corresponding statements for subasymptotic martingale and superasymptotic martingale.

Proof

since $\{X_n, \mathcal{F}_n, n\geq 1\}$is asymptotic martingale , for every $\varepsilon > 0$, there exsit $t_0 \in T$ such that for every $t, \sigma \in T$, $t,\sigma \geq t_0$,

Then $\|\int X_t\, dp - \int X_\sigma\, dp\| < 1\backslash 2\varepsilon \ldots\ldots\ldots (2.6.1)$

since $\{Y_n, \mathcal{F}_n, n\geq 1\}$is asymptotic martingale,

for every $\varepsilon > 0$, there exsit $t_0 \in T$ such that for every $t, \sigma \in T$, $t,\sigma \geq t_0$,

Then $\|\int Y_t\, dp - \int Y_\sigma\, dp\| < 1\backslash 2\varepsilon \ldots\ldots\ldots (2.6.2)$

We want to prove that

$\|\int (X_t - Y_t)dp - \int (X_\sigma - Y_\sigma)dp\| < \varepsilon$

We have from (2.6.1) and (2.6.2) we get

$\|\int (X_t - Y_t)dp - \int (X_\sigma - Y_\sigma)dp\| = \|\int X_t\, dp - \int X_\sigma\, dp - \int Y_t\, dp + \int Y_\sigma\, dp\| \leq \|\int X_t\, dp - \int X_\sigma\, dp\| + \|\int Y_t\, dp - \int Y_\sigma\, dp\| < 1\backslash 2\varepsilon + < 1\backslash 2\varepsilon = \varepsilon$

Thus$\{X_n - Y_n, \mathcal{F}_n, n\geq 1\}$is asymptotic martingale

Theorem 2.7

If $\{aX_n, \mathcal{F}_n, n\geq 1\}$ and $\{bY_n, \mathcal{F}_n, n\geq 1\}$are asymptotic martingale ,then $\{aX_n + b\, Y_n, \mathcal{F}_n, n\geq 1\}$is asymptotic martingale

With corresponding statements for subasymptotic martingale and superasymptotic martingale.

Proof

Since $\{aX_n, \mathcal{F}_n, n\geq 1\}$is asymptotic martingale,

for every $\varepsilon > 0$, there exsit $t_0 \in T$ such that for every $t, \sigma \in T$, $t,\sigma \geq t_0$,

Then $|a|\, \|\int X_t\, dp - \int X_\sigma\, dp\| < 1\backslash 2\varepsilon \ldots\ldots (2.7.1)$

since $\{bY_n, \mathcal{F}_n, n\geq 1\}$is asymptotic martingale,

for every $\varepsilon > 0$, there exsit $t_0 \in T$ such that for every $t, \sigma \in T$, $t,\sigma \geq t_0$,

Then $|b|\, \|\int Y_t\, dp - \int Y_\sigma\, dp\| < 1\backslash 2\varepsilon \ldots\ldots (2.7.2)$

We want to prove that $\|\int (aX_t + bY_t)dp - \int (aX_\sigma + bY_\sigma)dp\| < \varepsilon$

We have from (2.7.1) and (2.7.2) we get

$\|\int (aX_t + bY_t)dp - \int (aX_\sigma + bY_\sigma)dp\| = \|\int aX_t\, dp - \int aX_\sigma\, dp + \int bY_t\, dp - \int bY_\sigma\, dp\| \leq |a|\|\int X_t\, dp - \int X_\sigma\, dp\| + |b|\|\int Y_t\, dp - \int Y_\sigma\, dp\| < 1\backslash 2\varepsilon + 1\backslash 2\varepsilon = \varepsilon$

Thus$\{aX_n + bY_n, \mathcal{F}_n, n\geq 1\}$is asymptotic martingale

Theorem 2.8

If $\{Y_n, \mathcal{F}_n\}$ is a sub asymptotic martingale ,then

$\{\alpha Y_n, \mathcal{F}_n\}$,for $\alpha < 0$ is a super asymptotic martingale

$\{\alpha Y_n, \mathcal{F}_n\}$,for $\alpha > 0$ is a sub asymptotic martingale.

Proof

for $\alpha < 0$, since $\{Y_n, \mathcal{F}_n\}$ is a sub asymptotic martingale,

for every $\varepsilon > 0$, there exsit $t_0 \in T$ such that for every $t, \sigma \in T$, $t,\sigma \geq t_0$,

then $\|\int Y_\sigma\, dp - \int Y_t\, dp\| \leq \varepsilon_1$.

That is implies

$|\alpha|\|\int Y_\sigma\, dp - \int Y_t\, dp\| = \|\int \alpha Y_t\, dp - \int \alpha Y_\sigma\, dp\| \leq |\alpha|\varepsilon_1 = \varepsilon$

That is $\{\alpha Y_n, \mathcal{F}_n\}$ is a super asymptotic martingale

for $\alpha > 0$, since $\{Y_n, \mathcal{F}_n\}$ is a sub asymptotic martingale,

for every $\varepsilon > 0$, there exsit $t_0 \in T$ such that for every $t, \sigma \in T$, $t,\sigma \geq t_0$,

then $\|\int Y_\sigma\, dp - \int Y_t\, dp\| \leq \varepsilon_1$.

That is implies

$|\alpha|\, \|\int Y_\sigma\, dp - \int Y_t\, dp\| = \|\int \alpha Y_\sigma\, dp - \int \alpha Y_t\, dp\| \leq |\alpha|\varepsilon_1 = \varepsilon$

That is $\{\alpha Y_n, \mathcal{F}_n\}$ is a sub asymptotic martingale.

Theorem 2.9

If $\{X_{1,n}, \mathcal{F}_n\},\ldots,\{X_{m,n}, \mathcal{F}_n\}$ are a super asymptotic martingale, then$\{\min(X_{1,n},\ldots,X_{m,n}), \mathcal{F}_n\}$ is a super asymptotic martingale.

Proof

We want to prove

$\|\int \min(X_{t1,n}, \ldots, X_{tm,n})dp - \int \min(X_{\sigma 1,n}, \ldots, X_{\sigma m,n})\, dp\| \leq \varepsilon$

Since $\{X_{1,n}, \mathcal{F}_n\},\ldots,\{X_{m,n}, \mathcal{F}_n\}$ are a super asymptotic martingale, for every $\varepsilon > 0$, there exsit $t_0 \in T$ such that for every $t, \sigma \in T$, $t,\sigma \geq t_0$,

That is $\|\int X_{t1,n}dp - \int X_{\sigma 1,n}dp\| \leq \varepsilon$,..., $\|\int X_{tm,n}dp - \int X_{\sigma m,n}dp\| \leq \varepsilon$

Consider

$\|\int \min(X_{t1,n}, \ldots, X_{tm,n})dp - \int \min(X_{\sigma 1,n}, \ldots, X_{\sigma m,n})\, dp\| \leq \|\int X_{t1,n}dp - \int X_{\sigma 1,n}dp\| \leq \varepsilon \ldots\ldots (2.9.1)$

$\|\int \min(X_{t1,n},\dots,X_{tm,n})dp - \int \min(X_{\sigma1,n},\dots,X_{\sigma m,n})dp\| \le \|\int X_{t2,n}dp - \int X_{\sigma2,n}dp\| \le \varepsilon \ldots\ldots (2.9.2)$

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$\|\int \min(X_{t1,n},\dots,X_{tm,n})dp - \int \min(X_{\sigma1,n},\dots,X_{\sigma m,n})dp\| \le \|\int X_{tm,n}dp - \int X_{\sigma m,n}dp\| \le \varepsilon \ldots\ldots (2.9.m)$

From (2.9.1),…, (2.9.m) we obtain

$\|\int \min(X_{t1,n},\dots,X_{tm,n})dp - \int \min(X_{\sigma1,n},\dots,X_{\sigma m,n})dp\| \le \varepsilon$

Theorem 2.10

If $\{X_{1,n},\mathcal{F}_n\},\dots,\{X_{m,n},\mathcal{F}_n\}$ are a sub asymptotic martingale ,then $\{\min(X_{1,n},\dots,X_{m,n}),\mathcal{F}_n\}$ is a sub asymptotic martingale.

Proof

We want to prove

$\|\int \min(X_{\sigma1,n},\dots,X_{\sigma m,n})dp - \int \min(X_{t1,n},\dots,X_{tm,n})dp\| \le \varepsilon$

Since $\{X_{1,n},\mathcal{F}_n\},\dots,\{X_{m,n},\mathcal{F}_n\}$ are a sub asymptotic martingale for every $\varepsilon > 0$, there exsit $t_0 \in T$ such that for every $t,\sigma \in T$, $t,\sigma \ge t_0$,

That is $\|\int X_{\sigma1,n}dp - \int X_{t1,n}dp\| \le \varepsilon$,…, $\|\int X_{\sigma m,n}dp - \int X_{tm,n}dp\| \le \varepsilon$

Consider

$|\int \min(X_{\sigma1,n},\dots,X_{\sigma m,n})dp - \int \min(X_{t1,n},\dots,X_{tm,n})dp\| \le \|\int X_{\sigma1,n}dp - \int X_{t1,n}dp\| \le \varepsilon \ldots\ldots(2.10.1)$

$\|\int \min(X_{\sigma1,n},\dots,X_{\sigma m,n})dp - \int \min(X_{t1,n},\dots,X_{tm,n})dp\| \le \|\int X_{\sigma2,n}dp - \int X_{t2,n}dp\| \le \varepsilon \ldots\ldots(2.10.2)$

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$\|\int \min(X_{\sigma1,n},\dots,X_{\sigma m,n})dp - \int \min(X_{t1,n},\dots,X_{tm,n})dp\| \le \|\int X_{\sigma m,n}dp - \int X_{tm,n}dp\| \le \varepsilon \ldots\ldots(2.10.m)$

From (2.10.1),…, (2.10.m) we obtain

$\|\int \min(X_{\sigma1,n},\dots,X_{\sigma m,n})dp - \int \min(X_{t1,n},\dots,X_{tm,n})dp\| \le \varepsilon$

Theorem 2. 11

If $\{X_n,\mathcal{F}_n,n\ge1\}$ and $\{Y_n,\mathcal{F}_n,n\ge1\}$ are asymptotic martingale and $|X_t| = |Y_\sigma| = 1$, then $\{X_nY_n,\mathcal{F}_n,n\ge1\}$ is asymptotic martingale

With corresponding statements for subasymptotic martingale and superasymptotic martingal .

Proof

Since $\{X_n,\mathcal{F}_n,n\ge1\}$ is asymptotic martingale , for every $\varepsilon > 0$, there exsit $t_0 \in T$ such that for every $t,\sigma \in T$, $t,\sigma \ge t_0$, then $\|\int X_t\,dp - \int X_\sigma\,dp\| < 1\backslash 2\varepsilon$

Since $\{Y_n,\mathcal{F}_n,n\ge1\}$ is asymptotic martingale , for every $\varepsilon > 0$, there exsit $t_0 \in T$ such that for every $t,\sigma \in T$, $t,\sigma \ge t_0$,then $\|\int Y_t\,dp - \int Y_\sigma\,dp\| < 1\backslash 2\varepsilon$

We have to prove

$\|\int X_tY_t\,dp - \int X_\sigma Y_\sigma\,dp\| < \varepsilon$

$\|\int X_tY_t\,dp - \int X_\sigma Y_\sigma\,dp\| = \|\int X_tY_t\,dp - \int X_tY_\sigma\,dp + \int X_tY_\sigma\,dp - \int X_\sigma Y_\sigma\,dp\|$

$= \|X_t(\int Y_t\,dp - \int Y_\sigma\,dp) + Y_\sigma(\int X_t\,dp - \int X_\sigma\,dp)\|$

$\le |X_t|\|\int Y_t\,dp - \int Y_\sigma\,dp\| + |Y_\sigma|\ \|\int X_t\,dp - \int X_\sigma\,dp\|$

$< |X_t|1\backslash 2\varepsilon + |Y_\sigma|1\backslash 2\varepsilon = 1\backslash 2\varepsilon\,(|X_t| + |Y_\sigma|)$

$= 1\backslash 2\varepsilon\,(2) = \varepsilon$

Thus $\{X_nY_n,\mathcal{F}_n,n\ge1\}$ is asymptotic martingale.

Example 2.12

Let $(X_n,n\ge1)$ be an asymptotic martingale for increasing family $(\mathcal{F}_n, n\ge1)$ of σ -field.Then for all $n\ge1$, $Y_n=\sum_{k=1}^{n} X_n$ is an asymptotic martingale for increasing family $(\mathcal{F}_n, n\ge1)$ of σ –field.

solution

for every $\varepsilon > 0$, there exsit $t_0 \in T$ such that for every $t,\sigma \in T$,

$t,\sigma \ge t_0$, Then

$\|\int Y_t\,dp - \int Y_\sigma\,dp\| = \|\int \sum_{k=1}^{t} X_t\,dp - \int \sum_{k=1}^{\sigma} X_\sigma\,dp\| = \|\int (X_1 + X_2 + \dots + X_t)\,dp - \int (X_1 + X_2 + \dots + X_\sigma)\,dp\| = \|\int X_1dp + \int X_2dp + \dots + \int X_tdp - \int X_1dp - \int X_2dp - \int X_\sigma dp\|$

$= \|\int X_t\,dp - \int X_\sigma\,dp\|$

Since $(X_n,\mathcal{F}_n,n\ge1)$ is asymptotic martingale

$\|\int X_t\,dp - \int X_\sigma\,dp\| < \varepsilon$

Thus $(Y_n,\mathcal{F}_n,n\ge1)$ is an asymptotic martingale.

Lemma 2.13[2]

Let $(X_n ,n\ge1)$ be an asymptotic martingale for $(\mathcal{F}_n, n\ge1)$. Then $(\int Z_t)_{t\in T}$ is bounded.

proposition 2.14[2]

Let $\{X_n ,n\ge1\}$ and $\{Y_n,n\ge1\}$ be sequences adapted to $\{F_n, n\ge1\}$ .assume in addition that they are L^1-bounded.Then if $(\int X_t)_{t\in T}$ and $(\int Y_t)_{t\in T}$ are bounded then $(\int X_t \vee Y_t)_{t\in T}$ and $(\int X_t \wedge Y_t)_{t\in T}$ are bounded.

proposition 2.15[2]

If $\{X_n,\mathcal{F}_n,n\ge1\}$ and $\{Y_n,\mathcal{F}_n,n\ge1\}$ are asymptotic martingale,then

$\{X_n \vee Y_n,\mathcal{F}_n,n\ge1\}$ is asymptotic martingale.

Proof

We have to prove that $X_n \vee Y_n$ is asymptotic martingale. Write $Z_n = X_n \vee Y_n$ .By lemma(2.13) , $(\int X_t)$ and $(\int Y_t)$ are bounded. By proposition(2.14) , $(\int Z_t)_{t\in T}$ is bounded. let $\varepsilon > 0$ be given we can choose $t_0 \in T$ such if $t,\sigma \ge t_0$, then

$|\int X_\sigma - \int X_t| < \varepsilon$, $|\int Y_\sigma - \int Y_t| < \varepsilon \ldots\ldots. (2.15.1)$

Since, $(\int Z_t)_{t\in T}$ is bounded. we can choose $t_1 \ge t_0$ such that if $\sigma \ge t_0$, then $\int Z_\sigma \le \int Z_{t1} + \varepsilon \ldots\ldots.(2.15.2)$

Now given any bounded stopping time $\sigma \ge t_1$, let $\{X_{t1} < Y_{t1}\}$ and define $\sigma_1 \in T$ by

$\sigma_1 = t_1$ on A

$= \sigma$ on A^c.

Then $\int X_{t1} = \int A^c Z_{t1} + \int A\ X_{t1}$ ……(2.15.3)

$\int X_{\sigma1} = \int A^c X_\sigma + \int A\ X_{t1}$ ……(2.15.4)

Subtracting(1.15.4) from (1.15.3) ,then using (1.15.1) , we have

$\int A^c Z_{t1} = \int A^c X_\sigma + \int X_{t1} - \int X_{\sigma1} \le \int A^c Z_\sigma + \varepsilon$ ..(2.15.5)

Again $\int Y_{t1} = \int A\ Z_{t1} + \int A^c\ Y_{t1}$………(2.15.3')

$\int Y_{\sigma 1}=\int_{A} Y_{\sigma}+A^{c} Y_{t1}$…………(2.15.4')

Subtracting (2.15.4') from (2.15.3'),then using (2.15.1),we have

$\int_A Z_{t1}=\int_A Y_\sigma +\int Y_{t1}-\int Y_{\sigma 1} \leq \int_A Z_\sigma +\varepsilon$ ……(2.15.5') combinig (2.15.5) and (2.15.5') we have

$\int Z_{t1} \leq \int Z_\sigma +2\varepsilon$.

This ,together with (2.15.2),yields

$|\int Z_\sigma - \int Z_t| < 2\varepsilon$ ………(2.15.6)

This shows that the net$(\int Z_t)_{t\in T}$ is chauchy,hence convergent.

Definition 2.16[3]

Let $(\Omega, \mathcal{F}, P)$ be a probability space ,$\{X_1, X_2, \dots\}$ a sequence of integrable random variable on $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$ an increasing sequence of sub σ -field of $\mathcal{F}$, X_n is assumed $\mathcal{F}_n$ – measurable that is X_n $:(\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

The sequence $\{X_n, \mathcal{F}_n\}$ is said to be a Martingale **if and only if** for all n=1,2,. . .,$E[X_{n+1}|\mathcal{F}_n]=X_n$ a.e and asub martingale **if and only if** $E[X_{n+1}|\mathcal{F}_n]\geq X_n$ a.s .; and supermartingale **if and only if** $E[X_{n+1}|\mathcal{F}_n]\leq X_n$ a.e.

Example 2.17

Let B^2_t be a Brownian motion we comput $E[B^2_t - t|\mathcal{F}_S]$ for t≥s.

$E[B^2_t - t|\mathcal{F}_S]= E[(B_t-B_S+B_S)^2 - t|\mathcal{F}_S]= E[(B_t-B_S+B_S)^2|\mathcal{F}_S]-t=$

$E[(B_t-B_S)^2|\mathcal{F}_S]+ E[B^2_S|\mathcal{F}_S]+ 2E[(B_t-B_S)B_S|\mathcal{F}_S]-t$

Using independence, $E[(B_t-B_S)^2|\mathcal{F}_S]= E(B_t-B_S)^2=t-s$

Of course $E[B^2_S|\mathcal{F}_S]= B^2_s$

In the last term we use propert {if y∈ $\mathcal{F}$, $E|XY| < \infty$, then $E(XY|\mathcal{F})=XE(Y|\mathcal{F})$}

$E[(B_t-B_S)B_S|\mathcal{F}_S]=B_S E[(B_t-B_S)|\mathcal{F}_S]= B_S E(B_t-B_S)=0$

Since B_t-B_S is independence of F_S

Thus $E[B^2_t - t|\mathcal{F}_S]= B^2_S+t-s-t= B^2_S-s$

Thus $B^2_t - t$ is a martingale.

Theorem 2.18[2]

Every martingale is asymptotic martingale .

Proof

The adapted sequence $(X_n)_{n\in D}$ is defined so that martingale **if and only if**

$\int|X_n|<\infty$ for all n and $E[X_n|F_m]=X_m$ for all n,m∈D with n≥m ,in particular $\int X_n=\int X_m$.

If $t\in\tau$, choose n∈D with n≥t.

Then $\int X_t=\sum_k \int_{(t=k)} X_k=\sum_k \int_{(t=k)} X_n=\int X_n$.Thus $(\int X_t)$ is constant, the sequence $(X_n)_{n\in D}$ is asymptotic martingale.

Theorem 2.19

Let $(X_n, n\geq 1)$be a martingale for increasing family $(F_n, n\geq 1)$of σ –field. Let $(\mathcal{G}_n, n\geq 1)$ be another increasing family of σ –field with $\mathcal{G}_n \subseteq F_n$ for all n∈N .Then $Y_n=E(X_n|\mathcal{G}_n)$ is an asymptotic martingale for $(\mathcal{G}_n, n\geq 1)$ and not martingale.

proof

we want to prove $(Y_n,\mathcal{G}_n, n\geq 1)$ is asymptotic martingale,

for every $\varepsilon > 0$, there exsit $t_0 \in T$ such that for every $t, \sigma \in T$, $t,\sigma \geq t_0$,

$$\|\int Y_t\,dp - \int Y_\sigma\,dp\|=\|\int E(X_t|\mathcal{G}_t)\,dp - \int E(X_\sigma|\mathcal{G}_\sigma)\,dp\|=\|\int E(E(X_{t+1}|F_t)|\mathcal{G}_t)\,dp - \int E(E(X_{\sigma+1}|F_\sigma)|\mathcal{G}_\sigma)\,dp\|$$

Since $\mathcal{G}_n \subseteq F_n$

$$\|\int E(X_{t+1}|\mathcal{G}_t)\,dp - \int E(X_{\sigma+1}|\mathcal{G}_\sigma)\,dp\| = \|\int X_t\,dp - \int X_\sigma\,dp\|$$ by theorem(1.17)

$$\|\int X_t\,dp - \int X_\sigma\,dp\| < \varepsilon$$

Thus $(Y_n,\mathcal{G}_n, n\geq 1)$is asymptotic martingale.

We want to prove $(Y_n,\mathcal{G}_n, n\geq 1)$is not martingale

$E(Y_{n+1}|\mathcal{G}_n)= E(E(X_{n+1}|\mathcal{G}_{n+1})|\mathcal{G}_n)=E(X_{n+1}|\mathcal{G}_n)= X_n\neq Y_n$

Theorem 2.20[4]

If $\{X_n, \mathcal{F}_n, n\geq 1\}$ and $\{Y_n, \mathcal{F}_n, n\geq 1\}$are martingale,then $\{X_nY_n, \mathcal{F}_n, n\geq 1\}$is martingale.

Theorem 2.21

If $\{X_n, \mathcal{F}_n, n\geq 1\}$ and $\{Y_n, \mathcal{F}_n, n\geq 1\}$are martingale, then $\{X_nY_n, \mathcal{F}_n, n\geq 1\}$is asymptotic martingale.

Proof

By **theorem (2.18)** and **theorem (2.20)**, then $\{X_nY_n, \mathcal{F}_n, n\geq 1\}$ is asymptotic martingale .

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حول المارتنكل المقارب الجزئي والخاص

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الملخص

في هذا البحث قدمنا صنف جديد من مفاهيم وتعاريف المارتنكل المقارب بنوعيه الجزئي والخاص وبرهنا بعض خصائص المارتنكل المقارب الجزئي والخاص.