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Complement Hopfian Module

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ABSTRACT

A module M is called comp. Hopfian if any epomorphism f ∈ End (M) . ker f is complement of N in M for some submodule N of M . In this study, some properties of comp. Hopfian modules are investigated with examples .

Introduction

Hopfian (groups, rings and modules) have been studied by some authours since 1960. In the cuoreal study, all rings with identity and all modules are unital left R-modules unless otherwise specified. "Recall that a module M is called Hopfian if any surjective f ∈ End (M) is an isomorphism "[1] . Note that any Noetherian module is Hopfian. In this article, we introduce comp.Hopfian module as a generalized of Hopfian module. Amodule M is called complement Hopfian (simply comp. Hopfian module) if for any epomorphism f ∈ End (M), ker f is complement of N in M for some submodule N of M . The main goal in this thesis is to study these concepts and give some of their basic properties , characterizations and examples. Moreover, the study tries to explain the relationships among them, and other classes of modules.

1. Basic properties of comp. Hopfian module

This section introduces the concept comp. Hopfian modules, gives examples and some basic properties of this concept. This section begins with the following definitions :

Definition 1.1

A module M is called a complement Hopfian module (simply comp. Hopfian module) if for any epomorphism f ∈ End (M) , kerf is a complement of N in M for some submodule N of M .

Example 1.2

Z_6 is a comp. Hopfian z-module but Z_8 is not comp.Hopfian Z-module .

Remark 1.3

Each Hopfian module is comp. Hopfian module .But the converse is not true in general , as we see in the following example .

Example 1.4

Let $M = Q^n = Q \oplus Q \ \dots$, As Z-module , then M is comp. Hopfian module but it is not Hopfian module .

The following propositions explain the converse of remark (1.3) hold in the class of Dedekind finite module .

Recall that a module M is called Dedekind finite if f∘g = I , implies that g∘f = I , for all f ,g ∈ End (M) .[2]

Proposition 1.5

Let M be a Dedekind finite comp. Hopfian R-module then M is Hopfian module .

Proof :

Let f :M→M be epimorphism , since M is comp.Hopfian module then there

exists g ∈ End(M) , such that f∘g =I , and since M is Dedekind finite of End_R(M)

implies g∘f = I , hence f is an injection , Therefore M is Hopfian module.

We can get the following result from remark (1.3) and proposition (1.5) .

Corollary 1.6

Let M be a Dedekind finite module. Then M is comp. Hopfian module if and only if M is Hopfian module.

Since simple module has no proper submodules. The study reaches the following results .

Proposition 1.7.

Each simple module is comp. Hopfian module .

Proof : Let M be a module , and Let f : M⟶M be an epimorphism. Since M is simple module, implies that ker f is trivial submodule, so Its complement submodule of M. Hence M is comp. Hopfian module.

The converse of proposition (1.7) is not true in general , as shown in the following example:

Example 1.8

Let Z_6 be a Z-module is comp. Hopfian Z-module, but not simple Z-module .

Before we give our next result , it is important to have the following Lemma .

Lemma 1.9

If M be a module then each direct summand submodule is complement submodule .

Proof : Let M be a module and Let N be a direct summand submodule of M. Thus N is closed submodule by [3, prop(1.14)]. So, since each closed submodule is complement submodule by [2 ,prop (1.4) p.18], implies that N is complement submodule of M .

Since each submodule of semi-simple module is a direct summand , get the following result .

Proposition 1.10

Every semi- simple module M is comp. Hopfian module .

Proof : The proof direct by Lemma (1.9) .

The converse omit it proposition (1.10) is not true in general, as shown in the following example :

Example 1.11

Consider the module M = $Z_8 \oplus Z_2$ as a Z-module . Let A = ((2,1)) be the submodule generated by (2,1) , A = {(2,1),(4,0),(6,1),(0,0)}. Then A is not a direct summand becouse not exisit any submodule N such that N+A = M and N⋂A = 0 . but it is complement in M becouse there exisist any submodule K which is maximial such that A⋂K = 0.

It is well known that each submodule of semi-simple module is a direct summand Thus, it is significant to focus on the following remark .

Remark : 1.12

Let M be a semi-simple module. Then any submodule of M is comp. Hopfian module.

2 The relation between comp. Hopfian Module and another Module

This section introduces the relationship between comp. Hopfian module and another modules . Now, this section tries to show the relationship between (semi Hopfian, ascending chain condition and pseudo Hopfian) module .

"Recall that M is calld semi Hopfian module if any surjective f ∈ End (M) has a direct summand kernel". [4]

" Clearly , any Hopfian module is semi Hopfian"[4] **.**

Proposition 2.1

Any semi Hopfian module is comp. Hopfian module .

Proof : Let M be a semi Hopfian module and let f : M⟶M be a surjective and ker f is a direct summand of M . Since any direct summand is complement submodule by [2, p. 17]. so kerf is complement submodule in M. Therefore M is comp. Hopfian module .

If M is an extending module, can get semi Hopfian module from comp. Hopfian module of propositian (2.1). "Recall that M is calld extending module if every submodule N of M is an essential in a direct summand of M. Equivalently every closed submodule is a direct summand of M" . [5]

Proposition 2.2

Let M be an extending module. Then each comp.Hopfian module is semi Hopfian module .

Proof : Let $f: M \rightarrow M$ be a surjective and ker f is complement of M , since M is an extending module then ker f is a direct summand in M .Therefore M is semi Hopfian module .

It is preferable to concentrate on the following result from proposition (2.1) and proposition (2.3).

C0rollary 2.3

Let M be an extending module. Then M is comp. Hopfian module if and only if M is semi Hopfian module .

"Recall that an R-module M is called ascending chain condition denoted by (ACC), or that M is Northerian, we mean an increasing sequence $N_1 \subseteq N_2 \subseteq N_3 \ldots \subseteq M,$ of submodule of M ,then there exists a positive integer n such that $N_n = N_{n+1} = N_{n+2} = \cdots$ " . [6]

Proposition 2.4

If M has ascending chain condition on non–complement submodules . Then M is comp. Hopfian module .

Proof : Let g : M ⟶ M is a surjective and ker$g$ is a non–complement of M . So $ker\ g \subseteq Ker\ g^2 \subseteq Ker\ g^3 \subseteq \cdots$ be an ascend chain of non–complement submodules of M . By theory [7,prop(3.2.9), p.33] there will be a number n such as $ker\ (g^n) = Ker\ (g^{n+1})$, Now we claim that $Kerg = 0$, let $x \in kerg$ such that $g(x) = 0$. Since g is surjective , $g(a_1) = x$ for some $a_1 \in M$, also , $g(a_2) = a_1$ for some $a_2 \in M$. By repeating this argument , we have $g(a_n) = a_{n-1}$ for some $a_n \in M$.

Then

$g(a_1) = g^2\ (a_2) = \cdots = g^n\ (a_n) = x$. $hence\ g(x) = g^{n+1}(a_n)$ implies that $a_n \in Ker\ (g^{n+1}) = \ker(g^n)$. The result, $x = 0$. So have a contradiction. Therefore, M is comp. Hopfian module .

Proposition 2.5

If M/N is comp. Hopfian module for any non-complement submodule N of a module M . Then M is comp. Hopfian module .

Proof : Assume M is not comp. Hopfian module . Then there exist a surjection endomorphism $g : M \to M$ such that Ker$g$ is not a complement sudmodule of M .

But by assumption M/Ker$g \cong$ M is comp. Hopfian , this contradiction . So, M is comp. Hopfian module .

Conclusion

The main results are as follows

Each Hopfian module is comp. Hopfian module, while the converse is not true in general (see Remark ((1.3) and Example (1.4)) . and the converse is true under Dedekind finite conditions (see proposition (1.5)), every simple module is comp. Hopfian module, but the converse is not true in general (see proposition (1.7) and Example (1.8)), every semi-simple module is comp. Hopfian module, while the converse is not true in general (see proposition (1.10) and Example (1.11)), any semi Hopfian module is comp. Hopfian module, but the converse is not true ingeneral (see proposition (2.1)). And the converse is true under extending module (see proposition (2.2)), and we get if M has ascending chain condition on non-complement submodule. Then M is comp. Hopfian module (see proposition 2.4) .

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المقاسات الهوفينيه المكملة

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الملخص

المقاس M يسمى مقاسا هوفينيا مكمل اذا كان كل f ∈ End (M) , f داله شامله فان ker f مقاسا جزئيا مكمل في M . في هذا العمل سنعطي بعض القضايا المتعلقة بهذا المقاس مستشهدا بعض الامثلة.