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Applications of Nano Penta ($\mathcal{N}_{\mathcal{P}}$) Separation axioms using $\mathcal{N}_{\mathcal{P}}$_ open sets

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ABSTRACT

The main goal of this paper is to use the concept $\mathcal{N}_{\mathcal{P}}$_ open sets to present new classes of separation axioms in $\mathcal{N}_{\mathcal{P}}$ _topological spaces. Those new classes are $T_i^{\mathcal{N}P}$_spaces, i=0,1,2. We have studied some basic properties of these spaces. We also discussed the relationship between $T_i^{\mathcal{N}P}$ _spaces and Nano separation axioms ($\mathcal{N}T_i$_ spaces), i =0,1,2. Furthermore the paper deals with the relationship between the separation axioms throughout kernel set associated with the closed set which used to prove some theorems related to it. The hereditary and topological properties were also discussed.

1. Introduction

In 2013 Thivagar L.el at [1,2] introduced the idea of a Nano topological space with respect to a subset $\mathbb{X}$ of universe set $\mathcal{M}$, where $\mathfrak{R}$ an equivalence relation on $\mathcal{M}$. The pair $(\mathcal{M},\mathfrak{R})$ is known as the approximation space. So the lower approximation of $\mathbb{X}$ with respect to $\mathfrak{R}$ denoted by $\mathcal{L}_{\mathfrak{R}}(\mathbb{X}) = \cup_{x\in\mathcal{M}}\{[x]:\mathfrak{R}(x)\subseteq\mathbb{X}\}$and the upper approximation of $\mathbb{X}$ with respect to $\mathfrak{R}$ denoted by $U_{\mathfrak{R}}(\mathbb{X})$=$\cup_{x\in\mathcal{M}}\{[x]:\mathfrak{R}(x)\cap\mathbb{X}\neq\emptyset\}$, in which the boundary region of $\mathbb{X}$ with respect to $\mathfrak{R}$ is denoted by $B_{\mathfrak{R}}(\mathbb{X}) = U_{\mathfrak{R}}(\mathbb{X}) - \mathcal{L}_{\mathfrak{R}}(\mathbb{X})$. The elements of Nano topological space are called a Nano open sets. Thivagar also defined Nano closed set, Ncl set, Nint set, and also established Nano continuity maps, Nano open (Nano closed) maps and Nano home. and by using Nano open sets, the separation axioms ($\mathcal{N}T_i$_spaces) were known in 2019[3,4]. Through this concept they defined and investigated several topological properties. Topologists have focused their research on different types of class separation axioms [5]. Yaseen, R. et al [7] in 2021 studied the properties of Penta open sets in Penta topological spaces and used the concept of the $\mathcal{N}_{\mathcal{P}}$_topology introduced by Yaseen, R. et al [6]. They showed some practical examples of the $\mathcal{N}_{\mathcal{P}}$_ topology in real life [8]. In this paper, we use $\mathcal{N}_{\mathcal{P}}$_open and $\mathcal{N}_{\mathcal{P}}$_ closed sets defined by Yaseen R. et al [6] to present the concept of $\mathcal{N}_{\mathcal{P}}$_ Separation axioms on $\mathcal{N}_{\mathcal{P}}$_topological spaces which called $T_i^{\mathcal{N}_{\mathcal{P}}}$_spaces, where i=0,1,2. Moreover, some of its basic properties have been studied and the hereditary and topological properties were also discussed. Throughout the present paper, the spaces $\mathcal{M}$, $\widehat{\mathcal{M}}$ always means an $\mathcal{N}_{\mathcal{P}}$_topological spaces.

Definition 1.1.[6] Let $\mathcal{M}$ be a non_ empty universe set together with five Nano topologies $\mathfrak{I}_{\mathfrak{R}1}(\mathbb{X}), \mathfrak{I}_{\mathfrak{R}2}(\mathbb{X})$, $\mathfrak{I}_{\mathfrak{R}3}(\mathbb{X}), \mathfrak{I}_{\mathfrak{R}4}(\mathbb{X})$ and $\mathfrak{I}_{\mathfrak{R}5}(\mathbb{X})$ on $\mathcal{M}$ with respect to $\mathbb{X}$. We call $(\mathcal{M},\mathfrak{I}_{\mathfrak{R}\mathcal{P}}(\mathbb{X}))$ is a $\mathcal{N}_{\mathcal{P}}$_topological space with respect to $\mathbb{X}$,where $\mathfrak{I}_{\mathcal{P}} = (\tau_1,\tau_2,\tau_3,\tau_4,\tau_5)$.

A subset $\mathbb{A}$ is said to be Nano Penta open ($\mathcal{N}_{\mathcal{P}}$-open) set , if $\mathbb{A}\in(\mathfrak{I}_{\mathfrak{R}1}(\mathbb{X})\cup\mathfrak{I}_{\mathfrak{R}2}(\mathbb{X})\cup\mathfrak{I}_{\mathfrak{R}3}(\mathbb{X})\cup\mathfrak{I}_{\mathfrak{R}4}(\mathbb{X})\cup\mathfrak{I}_{\mathfrak{R}5}(\mathbb{X})$) and complement is said to be $\mathcal{N}_{\mathcal{P}}$-closed. So these $\mathcal{N}_{\mathcal{P}}$-open sets satisfies all the axioms of Nano topology $\mathcal{M}$.

2. $T_i^{\mathcal{N}P}$_spaces

In this section, we introduce $T_i^{\mathcal{N}P}$_spaces, where i=0,1,2 in $\mathcal{N}_{\mathcal{P}}$_topological spaces and study its properties.

Definition 2.1. A space $(\mathcal{M}, \Im_{\mathfrak{R}P}(\mathbb{X}))$ is said to be:

i - $T_0^{\mathcal{N}P}$_space according to the points v, u of $\mathcal{M}$ and there exists a $\mathcal{N}_{\mathcal{P}}$_open set containing one of them but not the other.

ii- $T_1^{\mathcal{N}P}$_space according to the points v, u of $\mathcal{M}$ and there exists two $\mathcal{N}_{\mathcal{P}}$ _open sets containing one of the two points but not the other.

iii- $T_2^{\mathcal{N}P}$_space according to the points v, u of $\mathcal{M}$ and there exists two distinct $\mathcal{N}_{\mathcal{P}}$_open sets $\mathbb{H}$ and $\mathbb{D}$, such that $v \in \mathbb{H}, u \in \mathbb{D}$.

Results 2.2. A Nano Penta Topological Spaces $(\mathcal{M}, \Im_{\mathfrak{R}P}(\mathbb{X}))$ is:

First case: $T_0^{\mathcal{N}P}$_space, if all the five Nano topologies is $\mathcal{N}T_0$ −spaces.

Second case: $T_i^{\mathcal{N}P}$_space, if there exist at least one of the five Nano topologies not $\mathcal{N}T_i$_space. where $i = 0,1,2$.

Third case: If there is no one of the five topologies is $\mathcal{N}T_i$_space, but the $\mathcal{N}_{\mathcal{P}}$_ topological space is $T_i^{\mathcal{N}P}$_space. where $i = 1,2$.

To explain these cases, we are going to discuss the following three examples

Examples 2.3.

Let $\mathcal{M} = \{a, b, c, d, f\}$ on $\mathcal{M}/\mathfrak{R}=\{\{a\}, \{b, d\}, \{c\}, \{f\}\}$ with

Let $\mathbb{X}_1 = \{c, f\} \subseteq \mathcal{M} \rightarrow \Im_{\mathfrak{R}1}(\mathbb{X}) = \{\mathcal{M}, \emptyset, \{c, f\}\}$

Let $\mathbb{X}_2 = \{a\} \subseteq \mathcal{M} \rightarrow \Im_{\mathfrak{R}2}(\mathbb{X}) = \{\mathcal{M}, \emptyset, \{a\}\}$

Let $\mathbb{X}_3 = \{a, c, f\} \subseteq \mathcal{M} \rightarrow \Im_{\mathfrak{R}3}(\mathbb{X}) = \{\mathcal{M}, \emptyset, \{a, c, f\}\}$

Let $\mathbb{X}_4 = \{f\} \subseteq \mathcal{M} \rightarrow \Im_{\mathfrak{R}4}(\mathbb{X}) = \{M, \emptyset, \{f\}\}$

Let $\mathbb{X}_5 = \{a, f\} \subseteq \mathcal{M} \rightarrow \Im_{\mathfrak{R}5}(\mathbb{X}) = \{\mathcal{M}, \emptyset, \{a, f\}\}$

Hence: $\Im_{\mathfrak{R}\mathcal{P}}(\mathbb{X}) = \{\mathcal{M}, \emptyset, \{a, f\}, \{a, c, f\}, \{a\}, \{c, f\}, \{f\}\}$. Then $\mathcal{M}$ is $T_0^{\mathcal{N}P}$_space for {a} and {c,f} with the five Nano topologies is $\mathcal{N}T_0$ −space. Also $\mathcal{M}$ is $T_1^{\mathcal{N}P}(T_2^{\mathcal{N}P})$_space for {a} and {f} with the five Nano topologies is not $\mathcal{N}T_1(\mathcal{N}T_2)$space.

1. $\mathcal{M} = \{a, b, c, d\}$ on $\mathbb{X} = \{a, b, c\}$ with $\mathcal{M}/\mathfrak{R}_1$= { {a},{b,c},{d}},
$\mathcal{M}/\mathfrak{R}_2$= { {a,d},{b,c}} , $\mathcal{M}/\mathfrak{R}_3$={{a,b,d},{c}}, $\mathcal{M}/\mathfrak{R}_4$= { {a,b},{c},{d}},
$\mathcal{M}/\mathfrak{R}_5$= {{c,d},{a,b}}. We can obtain the following results

	$\mathcal{M}/\mathfrak{R}_1$	$\mathcal{M}/\mathfrak{R}_2$	$\mathcal{M}/\mathfrak{R}_3$	$\mathcal{M}/\mathfrak{R}_4$	$\mathcal{M}/\mathfrak{R}_5$
$\mathcal{L}_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$	{a,b,c}	{b,c}	{c}	{a,b,c}	{a, b}
$U_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$	{a,b,c}	$\mathcal{M}$	$\mathcal{M}$	{a,b,c}	$\mathcal{M}$
$B_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$	$\emptyset$	{a,d}	{a, b, d}	$\emptyset$	{c,d}

Hence:

$\underline{\sum_{\mathcal{P}=1}^{5} \mathfrak{R}_{\mathcal{P}}}(\mathbb{X})$ ={a,b,c} and $\overline{\sum_{\mathcal{P}=1}^{5} \mathfrak{R}_{\mathcal{P}}}(\mathbb{X}) = \{a, b, c\}$.

So, we will be on

$\underline{\sum_{\mathcal{P}=1}^{5} \mathfrak{R}_{\mathcal{P}}}(\mathbb{X}) = \overline{\sum_{\mathcal{P}=1}^{5} \mathfrak{R}_{\mathcal{P}}}(\mathbb{X}) = \{a, b, c\}$, then $\Im_{\sum_{\mathcal{P}=1}^{5} \mathfrak{R}_{\mathcal{P}}}(\mathbb{X}) = \{\mathcal{M}, \emptyset, \{a, b, c\}\}$.

Then $\mathcal{M}$is $T_0^{\mathcal{N}P}$_space for {a,b,c} and {d} with the five Nano topologies is $\mathcal{N}T_0$ −space. But not $T_1^{\mathcal{N}P}(T_2^{\mathcal{N}P})$_space for {a,b,c} and {d}with there exist at least one of five Nano topologies is $\mathcal{N}T_1(\mathcal{N}T_2)$_space.

3. $\mathcal{M}$ = {a,b,c,d,e} , $\mathbb{X} \subseteq \mathcal{M}$,with $\mathcal{M}/\mathfrak{R}_1$={{a,b},{c,d},{e}},
$\mathcal{M}/\mathfrak{R}_2$={{a,e},{b},{d,c}}, $\mathcal{M}/\mathfrak{R}_3$={{b,c,d},{a,e}}, $\mathcal{M}/\mathfrak{R}_4$= {{b,c},{a},{d,e}}, $\mathcal{M}/\mathfrak{R}_5$= {{d,e},{a,b,c}}, We get

$\mathcal{P} = 1,2,3,4,5$	$\mathbb{X}_{\mathcal{P}}$	$\mathcal{L}_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$	$U_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$	$B_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$	$\Im_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$
$\mathcal{M}/\mathfrak{R}_1$	{a,b,c}	{a,b}	{a, b, c, d}	{c,d}	$\{\mathcal{M}, \emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$
$\mathcal{M}/\mathfrak{R}_2$	{b}	{b}	{b}	$\emptyset$	$\{\mathcal{M}, \emptyset, \{b\}\}$
$\mathcal{M}/\mathfrak{R}_3$	{b,c,d}	{b,c,d}	{b,c,d}	$\emptyset$	$\{\mathcal{M}, \emptyset$,{b,c,d}}
$\mathcal{M}/\mathfrak{R}_4$	{b,c}	{b,c}	{b,c}	$\emptyset$	$\{\mathcal{M}, \emptyset, \{b, c\}\}$
$\mathcal{M}/\mathfrak{R}_5$	{a,b}	$\emptyset$	{a,b,c}	{a,b,c}	$\{\mathcal{M}, \emptyset$,{a,b,c}}

Hence:

$\Im_{\mathfrak{R}\mathcal{P}}(\mathbb{X}) = \{\mathcal{M}, \emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}, \{b\}, \{b, c, d\}, \{b, c\}, \{a, b, c\}\}$ .Then $\mathcal{M}$is $T_0^{\mathcal{N}P}$_space for {b} and {c, d} with the five Nano topologies is $\mathcal{N}T_0$ −space. Also $\mathcal{M}$is $T_1^{\mathcal{N}P}(T_2^{\mathcal{N}P})$_space for {b} and {c d} with there exist at least one of five Nano topologies is $\mathcal{N}T_1(\mathcal{N}T_2)$ −space.

Theorem 2.4.

A space $(\mathcal{M}, \Im_{\mathfrak{R}P}(\mathbb{X}))$, is said to be $T_0^{\mathcal{N}P}$_space according to the points v, u of $\mathcal{M}$, either $v \in U_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$ and $u \in (U_{\mathfrak{R}\mathcal{P}}(\mathbb{X}))^c$ or $v \in \mathcal{L}_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$ and $u \in B_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$, if $U_{\mathfrak{R}\mathcal{P}}(\mathbb{X}) \neq \mathcal{M}$, $\mathcal{L}_{\mathfrak{R}\mathcal{P}}(\mathbb{X}) \neq \emptyset$, $\ni U_{\mathfrak{R}\mathcal{P}}(\mathbb{X}) \neq \mathcal{L}_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$.

Proof.

In this case $\Im_{\mathfrak{R}\mathcal{P}}(\mathbb{X}) = \{\mathcal{M}, \emptyset, \mathcal{L}_{\mathfrak{R}\mathcal{P}}(\mathbb{X}), U_{\mathfrak{R}\mathcal{P}}(\mathbb{X}), B_{\mathfrak{R}\mathcal{P}}(\mathbb{X})\}$, suppose that $v \in U_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$ and $u \in (U_{\mathfrak{R}\mathcal{P}}(\mathbb{X}))^c$. Then, $U_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$ is $\mathcal{N}_{\mathcal{P}}$_O. set containing one of them but not the other. That is $\mathcal{M}$ is $T_0^{\mathcal{N}P}$_space for $v$and u, or suppose that $v \in \mathcal{L}_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$ and $u \in B_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$. since $\mathcal{L}_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$ is $\mathcal{N}_{\mathcal{P}}$_O. set containing v and $u \notin \mathcal{L}_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$. Then $\mathcal{M}$ is $T_0^{\mathcal{N}P}$_space for $v$and u.

Corollary 2.5.

A space $(\mathcal{M}, \Im_{\mathfrak{R}P}(\mathbb{X}))$ is $T_0^{\mathcal{N}P}$_space according to v, u of $\mathcal{M}$, such that $v \in \mathcal{L}_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$ and $u \in B_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$ if $U_{\mathfrak{R}\mathcal{P}}(\mathbb{X}) = \mathcal{M}$ and $\mathcal{L}_{\mathfrak{R}\mathcal{P}}(\mathbb{X}) \neq \emptyset$.

Corollary 2.6.

A space $(\mathcal{M}, \Im_{\mathfrak{R}P}(\mathbb{X}))$ is $T_0^{\mathcal{N}P}$_space according to v, u of $\mathcal{M}$, such that $v \in \Im_{\mathfrak{R}}(\mathbb{X})$ and $u \in (U_{\mathfrak{R}\mathcal{P}}(\mathbb{X}))^c$, if $U_{\mathfrak{R}\mathcal{P}}(\mathbb{X}) = \mathcal{L}_{\mathfrak{R}\mathcal{P}}(\mathbb{X}) = \mathbb{X}$.

Theorem2.7.

If $(\mathcal{M}, \Im_{\mathfrak{R}}(\mathbb{X}))$ is a $\mathcal{N}T_0$_space, then $(\mathcal{M}, \Im_{\mathfrak{R}P}(\mathbb{X}))$ is a $T_0^{\mathcal{N}P}$_space.

Proof.

suppose $\mathcal{M}$ is $\mathcal{N}T_0$_space, ∃ a $\mathcal{N}$_O. set containing $\mathbb{H}$ one of them but not the other for $v, u \in \mathcal{M}$, $v \neq u$ such that $v \in \mathbb{H}$ and $u \notin \mathbb{H}$. As Every $\mathcal{N}$_O. sets is $\mathcal{N}_{\mathcal{P}}$_O. [2], hence $\mathbb{H}_{\mathcal{N}P}$, for $v, u \in \mathcal{M}$ such that $v \in \mathbb{H}_{\mathcal{N}P}$ and $u \notin \mathbb{H}_{\mathcal{N}P}$. Then $\mathcal{M}$ is $T_0^{\mathcal{N}P}$_space.

Theorem 2.8.

Any $\mathcal{N}T_i$_space is $T_i^{\mathcal{N}P}$_space. Where $i \in \{1,2\}$

Proof.

we prove that the theorem for $T_1^{\mathcal{N}P}$_space.

Let $\mathcal{M}$ be $\mathcal{N}T_1$_space, ∃ two $\mathcal{N}$_O. sets containing one of the two points, but not the other, for $v, u \in \mathcal{M}$, $v \neq u$, since every $\mathcal{N}$_O. sets is $\mathcal{N}_{\mathcal{P}}$_O. [2]. Then $\mathbb{H}, \mathbb{D}$ are $\mathcal{N}_{\mathcal{P}}$_O. sets such that $v \in \mathbb{H}$ and $u \notin \mathbb{H}$ or $v \notin \mathbb{D}$ and $u \in \mathbb{D}$. Then $\mathcal{M}$ is $T_1^{\mathcal{N}P}$_space.

Remark 2.9. The converse is not true. (As in second case of result 2)

Theorem2.10. Every $T_i^{\mathcal{N}P}$_space is $T_{i-1}^{\mathcal{N}P}$_space, where $i = 1,2$

Proof.

we prove that the theorem for $i = 1$

$\mathcal{M}$ $T_1^{\mathcal{N}P}$_space if for v, u of $\mathcal{M}$, there exists two $\mathcal{N}_{\mathcal{P}}$ _O. sets $\mathbb{H}$ containing one of the two points, but not the other. Furthermore if ∃ $\mathbb{H}$ is $\mathcal{N}_{\mathcal{P}}$_O., such that $v \in \mathbb{H}$ and $u \notin \mathbb{H}$, then $\mathcal{M}$ is $T_0^{\mathcal{N}P}$_space.

we prove that the theorem for $i = 2$

suppose that $\mathcal{M}$ is $T_2^{\mathcal{N}P}$_space, if for v, u of $\mathcal{M}$, then there exists two disjoint $\mathcal{N}_{\mathcal{P}}$ _O. sets containing one of the two points, but not the other. Then ∃ $\mathbb{H}, \mathbb{D}$ are $\mathcal{N}_{\mathcal{P}}$_O.such that $v \in \mathbb{H}$ and $u \notin \mathbb{H}$ or $u \in \mathbb{D}$ and $v \notin \mathbb{D}$. Then $\mathcal{M}$ is $T_1^{\mathcal{N}P}$_space.

Theorem 2.11.

A space $(\mathcal{M}, \mathfrak{I}_{\mathfrak{R}P}(\mathbb{X}))$ is said to be $T_1^{\mathcal{N}P}$_space for v, u of $\mathcal{M}$, when $v \in \mathcal{L}_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$ and $u \in B_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$, if $U_{\mathfrak{R}\mathcal{P}}(\mathbb{X}) = \mathcal{M}$ and $\mathcal{L}_{\mathfrak{R}\mathcal{P}}(\mathbb{X}) \neq \emptyset$ ∋ $\mathcal{L}_{\mathfrak{R}\mathcal{P}}(\mathbb{X}) \neq U_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$.

Proof.

Suppose $v \in \mathcal{L}_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$ and $u \in B_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$ for $v, u \in \mathcal{M}$, since $\mathcal{L}_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$ and $B_{\mathfrak{R}\mathcal{P}}(\mathbb{X})$ are distinct $\mathcal{N}_{\mathcal{P}}$_O. sets. Then $\mathcal{M}$ is $T_1^{\mathcal{N}P}$_space.

3. Properties of $\mathcal{N}_{\mathcal{P}}$_ Separation axioms via $\mathcal{N}_{\mathcal{P}}$_ kernel sets

In this section we extend the properties $\mathcal{N}_{\mathcal{P}}$_ Separation axioms on $\mathcal{N}_{\mathcal{P}}$_topological spaces throughout $\mathcal{N}_{\mathcal{P}}$_ kernel set associated with the $\mathcal{N}_{\mathcal{P}}$_ closed set.

Definition 3.1.

Let $(\mathcal{M}, \mathfrak{I}_{\mathfrak{R}P}(\mathbb{X}))$ and $(\breve{\mathcal{M}}, \mathfrak{I}_{\breve{\mathfrak{R}}P}(\breve{\mathbb{X}}))$ be a $\mathcal{N}_{\mathcal{P}}$_topological spaces with respect to $\mathbb{X}$ and $\breve{\mathbb{X}}$ respectively. A map $f: (\mathcal{M}, \mathfrak{I}_{\mathfrak{R}P}(\mathbb{X})) \to (\breve{\mathcal{M}}, \mathfrak{I}_{\breve{\mathfrak{R}}P}(\breve{\mathbb{X}}))$ is Nano Penta continuous ($Con_{\mathcal{N}_{\mathcal{P}}}$) if $f^{-1}(\mathbb{H})$ of each $\mathcal{N}_{\mathcal{P}}$_O. in $\breve{\mathcal{M}}$ is $\mathcal{N}_{\mathcal{P}}$_O.in $\mathcal{M}$.

Proposition3.2.

The $Con_{\mathcal{N}_{\mathcal{P}}}$-image of $T_0^{\mathcal{N}P}$_space is $T_0^{\mathcal{N}P}$_space.

Proof.

Suppose that $f: (\mathcal{M}, \mathfrak{I}_{\mathfrak{R}P}(\mathbb{X})) \to (\breve{\mathcal{M}}, \mathfrak{I}_{\breve{\mathfrak{R}}P}(\breve{\mathbb{X}}))$ and $d, e \in \breve{\mathcal{M}}$, since f is onto, then $\exists c, z$ such that $c = f^{-1}(d) \wedge z = f^{-1}(e)$. Since $\mathcal{M}$ is a $T_0^{\mathcal{N}P}$_space, then ∃ $\mathcal{N}_{\mathcal{P}}$_O. set containing one of the two points c and z but not the other, for $c \in \mathbb{H} \wedge z \notin \mathbb{H}$ or $\mathfrak{S}$ such that $z \in \mathfrak{S} \wedge c \notin \mathfrak{S}$, so $f^{-1}(c) \in f(\mathbb{H}) = \breve{\mathbb{H}}$ or $f^{-1}(z) \in f(\mathfrak{S}) = \breve{\mathfrak{S}}$, then $d \in \breve{\mathbb{H}} \wedge e \notin \breve{\mathbb{H}}$ or $e \in \breve{\mathfrak{S}} \wedge d \notin \breve{\mathfrak{S}}$. We get $\breve{\mathcal{M}}$ is a $T_0^{\mathcal{N}P}$_space.

Remark 3.3.

The image of a $T_i^{\mathcal{N}P}$_space under a $Con_{\mathcal{N}_{\mathcal{P}}}$ map need not be $T_i^{\mathcal{N}P}$ space. Where $i = 1,2$

Example 3.4.

Let $\mathfrak{I}_{\mathfrak{R}P}(\mathbb{X}) = \{\mathcal{M}, \emptyset, \{a,b\}, \{c,d\}, \{a,b,c,d\}, \{b\}, \{b,c,d\}, \{b,c\}, \{a,b,c\}\}$ on $\mathcal{M}$ = {a,b,c,d,e} and $\mathfrak{I}_{\sum_{\mathcal{P}=1}^{5} \mathfrak{R}_{\mathcal{P}}}(\dot{\mathbb{X}}) = \{\dot{\mathcal{M}}, \dot{\emptyset}, \{\dot{a}, \dot{b}, \dot{c}\}\}$ on $\dot{\mathcal{M}} = \{\dot{a}, \dot{b}, \dot{c}, \dot{d}\}$, so that $f: (\mathcal{M}, \mathfrak{I}_{\mathfrak{R}P}(\mathbb{X})) \to (\dot{\mathcal{M}}, \mathfrak{I}_{\dot{\mathfrak{R}}P}(\dot{\mathbb{X}}))$, define by $f(a) = \dot{d}$, $f(b) = \dot{b}$, $f(d) = \dot{a}$, $f(c) = \dot{c}$, $f(e) = \dot{e}$, then $\mathcal{M}$ is $T_i^{\mathcal{N}P}$_space and $f: \mathcal{M} \to \dot{\mathcal{M}}$ is $Con_{\mathcal{N}_{\mathcal{P}}}$ map, but not $T_i^{\mathcal{N}P}$_space, $i = 1,2$.

Proposition 3.5.

For every pair of distinct points $v, u \in \mathcal{M}$, $cl_{\mathcal{N}_{\mathcal{P}}}\{v\} \neq cl_{\mathcal{N}_{\mathcal{P}}}\{u\}$ iff a space $\mathcal{M}$ is $T_0^{\mathcal{N}P}$_space.

Proof.

For $v, u \in \mathcal{M}$, $v \neq u$ with $cl_{\mathcal{N}_{\mathcal{P}}}\{v\} \neq cl_{\mathcal{N}_{\mathcal{P}}}\{u\}$. Suppose that $w \in \mathcal{M}$ such that $w \in cl_{\mathcal{N}_{\mathcal{P}}}\{v\}$, $w \notin cl_{\mathcal{N}_{\mathcal{P}}}\{u\}$. Assume $v \notin cl_{\mathcal{N}_{\mathcal{P}}}\{u\}$. If $v \in cl_{\mathcal{N}_{\mathcal{P}}}\{u\}$ then $\{v\} \subseteq cl_{\mathcal{N}_{\mathcal{P}}}\{u\} \to cl_{\mathcal{N}_{\mathcal{P}}}\{v\} \subseteq cl_{\mathcal{N}_{\mathcal{P}}}\{u\}$. Thus $w \in cl_{\mathcal{N}_{\mathcal{P}}}\{v\} \wedge w \in cl_{\mathcal{N}_{\mathcal{P}}}\{u\}$ this is contradiction. Hence $\mathcal{M} \backslash cl_{\mathcal{N}_{\mathcal{P}}}\{u\}$ is $\mathcal{N}_{\mathcal{P}}$_O. set containing v, but not u. Then $\mathcal{M}$ is $T_0^{\mathcal{N}P}$_space. Conversely,

Suppose that $v, u \in \mathcal{M}$, $v \neq u$, since $\mathcal{M}$ is a $T_0^{\mathcal{N}P}$_space and $\mathbb{X} \in \mathcal{N}_{\mathcal{P}}O(\mathcal{M})$ such that $v \in \mathbb{X} \wedge u \notin \mathbb{X}$, therefore $cl_{\mathcal{N}_{\mathcal{P}}}\{u\} \subseteq \mathcal{M} \backslash \mathbb{X}$. Hence $v \in \mathcal{M} \backslash \mathbb{X}$ as $v \notin cl_{\mathcal{N}_{\mathcal{P}}}\{v\} \wedge u \in cl_{\mathcal{N}_{\mathcal{P}}}\{u\}$, then $cl_{\mathcal{N}_{\mathcal{P}}}\{v\} \neq cl_{\mathcal{N}_{\mathcal{P}}}\{u\}$.

Definition 3.6.

The intersection of all $\mathcal{N}_{\mathcal{P}}$_ O. subset of $(\mathcal{M}, \mathfrak{I}_{\mathfrak{R}P}(\mathbb{X}))$ containing $\mathbb{A}$ is called the Nano Penta Kernel of $\mathbb{A}$, in short $(\mathcal{K}_{\mathcal{N}_{\mathcal{P}}}(\mathbb{A}))$.

Remark 3.7. For any two subsets $\mathbb{A}, \mathbb{B}$ of a space $(\mathcal{M}, \mathfrak{I}_{\mathfrak{R}P}(\mathbb{X}))$,

1. $\mathbb{A} \subseteq \mathbb{B}$ implies $\mathcal{K}_{\mathcal{N}_{\mathcal{P}}}(\mathbb{A}) \subseteq \mathcal{K}_{\mathcal{N}_{\mathcal{P}}}(\mathbb{B})$, where $P = 1,2,3,4,5$.
2. $\mathcal{K}_{\mathcal{N}_{\mathcal{P}}}(\mathcal{K}_{\mathcal{N}_{\mathcal{P}}}(\mathbb{A})) = \mathcal{K}_{\mathcal{N}_{\mathcal{P}}}(\mathbb{A})$.

Theorem 3.8.

A space $\mathcal{M}$ is said to be $T_0^{\mathcal{N}P}$_space iff $\forall v, u \in \mathcal{M}$, $v \neq u$ either $u \notin \mathcal{K}_{\mathcal{N}_{\mathcal{P}}}\{v\}$ or $v \notin \mathcal{K}_{\mathcal{N}_{\mathcal{P}}}\{u\}$.

Proof.

Suppose that $\mathcal{M}$ is a $T_0^{\mathcal{N}P}$_space, $\forall v \neq u \in \mathcal{M}$, ∃ a $\mathcal{N}_{\mathcal{P}}$_O. set $\mathbb{H}$ containing one of them but not the other. Thus either $v \in \mathbb{H}$ and $u \notin \mathbb{H} \to u \notin \mathcal{K}_{\mathcal{N}_{\mathcal{P}}}\{v\}$ or $u \in \mathbb{H}$ and $v \notin \mathbb{H} \to v \notin \mathcal{K}_{\mathcal{N}_{\mathcal{P}}}\{u\}$.

Conversely, $\forall \mathcal{v} \neq \mathcal{u} \in \mathcal{M}$, either $\mathcal{u} \notin \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\}$ or $\mathcal{v} \notin \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}$ .Then $\mathcal{N}_\mathcal{p}$_O. set $\mathbb{H}$ containing one of them but not the other, thus $\mathcal{M}$ is a $T_0^{\mathcal{N}P}$_space.

Theorem 3.9.
A space $\mathcal{M}$ is said to be $T_1^{\mathcal{N}P}$_space iff $\forall \mathcal{v} \neq \mathcal{u} \in \mathcal{M}, \mathcal{u} \notin \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\}$ and $\mathcal{v} \notin \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}$.

Proof.
Let $\mathcal{M}$ be a $T_1^{\mathcal{N}P}$_space, $\mathcal{v}, \mathcal{u} \in \mathcal{M}, \mathcal{v} \neq \mathcal{u}$, then $\exists$ two $\mathcal{N}_\mathcal{p}$_O. sets $\mathbb{H}, \mathbb{D}$ containing one of the two points, but not the other such that $\mathcal{v} \in \mathbb{H}, \mathcal{u} \notin \mathbb{H}$ or $\mathcal{u} \in \mathbb{D}, \mathcal{v} \notin \mathbb{D}$.
Hence $\mathcal{u} \notin \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\}$ or $\mathcal{v} \notin \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}$.
Conversely, $\exists$ two $\mathcal{N}_\mathcal{p}$_O. sets containing one of the two points, but not the other, $\mathcal{v} \neq \mathcal{u} \in \mathcal{M}$, thus $\mathcal{u} \notin \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\}$ and $\mathcal{v} \notin \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}$. Then $\mathcal{M}$ is a $T_1^{\mathcal{N}P}$_space.

Corollary 3.10.
A space $\mathcal{M}$ is said to be $T_1^{\mathcal{N}P}$_space iff $\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} = \{\mathcal{u}\}, \forall \mathcal{u} \in \mathcal{M}$.

Proof.
Let $\mathcal{M}$ be a $T_1^{\mathcal{N}P}$_space and $\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} \neq \{\mathcal{u}\}$, then either $\mathcal{v} \in \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}$ or $\mathcal{u} \in \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\}$. Hence (by theorem3.9) $\mathcal{M}$ is not a $T_1^{\mathcal{N}P}$_space this is contradiction. So $\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} = \{\mathcal{u}\}$. Conversely,
Let $\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} = \{\mathcal{u}\}, \mathcal{u} \in \mathcal{M}$ and suppose that $\mathcal{M}$ is not a $T_1^{\mathcal{N}P}$_space. By (theorem3.9) we get $\mathcal{v} \in \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} \rightarrow \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} \neq \{\mathcal{u}\}$, this is contradiction. Thus $\mathcal{M}$ is a $T_1^{\mathcal{N}P}$_space.

Corollary 3.11.
A space $\mathcal{M}$ is said to be $T_1^{\mathcal{N}P}$_space iff $\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} \cap \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\} = \emptyset, \forall \mathcal{v} \neq \mathcal{u} \in \mathcal{M}$.

Proof.
Let $\mathcal{M}$ be a $T_1^{\mathcal{N}P}$_space, then $\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} = \{\mathcal{u}\}$ and $\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\} = \{\mathcal{v}\}$. By Corollary 3.10. Thus $\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} \cap \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\} = \emptyset$.
Conversely,
Let $\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} \cap \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\} = \emptyset, \mathcal{v} \neq \mathcal{u} \in \mathcal{M}$ and a space $\mathcal{M}$ be dose not $T_1^{\mathcal{N}P}$_space.
Then $\forall \mathcal{v} \neq \mathcal{u} \in \mathcal{M} \rightarrow \mathcal{v} \in \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}$ or $\mathcal{u} \in \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\}$, so $\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} \cap \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\} \neq \emptyset$, this is contradiction. Hence $\mathcal{M}$ be a $T_1^{\mathcal{N}P}$_space.

Lemma 3.12.
The $\mathcal{M}$ space is a $T_1^{\mathcal{N}P}$_space iff $\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} = cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}, \forall \mathcal{u} \in \mathcal{M}$.

Proposition 3.13.
Let $(\mathcal{M}, \mathfrak{I}_{\mathfrak{R}P}(\mathbb{X}))$ be a $\mathcal{N}_\mathcal{p}$_topological space. $\mathcal{u} \in cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\}$ iff $\mathcal{v} \in \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}, \forall \mathcal{u} \neq \mathcal{v} \in \mathcal{M}$.

Proof.
Let $\mathcal{v} \notin \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}, \mathcal{u} \neq \mathcal{v} \in \mathcal{M}$, then $\exists$ $\mathcal{N}_\mathcal{p}$_O. set $\mathbb{D}$ containing $\mathcal{u}$ suchthat $\mathcal{v} \notin \mathbb{D} \rightarrow \mathcal{u} \notin cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\}$, the converse can be proved in the similar way.

Proposition 3.14.
A space $\mathcal{M}$ is said to be $T_1^{\mathcal{N}P}$_space iff $\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} \subseteq \mathbb{H}$, $\forall$ $\mathcal{N}_\mathcal{p}$_C. $\mathbb{H}$ set and $\mathcal{u} \in \mathbb{H}$.

Proof.
Suppose that $\mathcal{M}$ is a $T_1^{\mathcal{N}P}$_space and $\mathbb{H}$ is $\mathcal{N}_\mathcal{p}$_C. set, $\mathcal{u} \in \mathbb{H}$. Then, $\mathcal{v} \in \mathbb{H}^c$ is $\mathcal{N}_\mathcal{p}$_O. set. Since $\mathcal{M}$ is a $T_1^{\mathcal{N}P}$_space. Then $cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\} \subseteq \mathbb{H}^c$, by Proposition 3.13, we get $\mathcal{u} \notin cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\} \rightarrow \mathcal{v} \notin \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}$.
Thus $\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} \subseteq \mathbb{H}$. Conversely, Let $\mathcal{N}_\mathcal{p}$_C. $\mathbb{H}$ set, $\mathcal{u} \in \mathbb{H}$. Then $\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} \subseteq \mathbb{H}$ and $\mathcal{u} \in \mathbb{H} \in \mathfrak{I}_{\mathfrak{R}\mathcal{P}}$, then $\mathcal{v} \in \mathbb{H}^c, \forall \mathcal{v} \notin \mathbb{H}$ is $\mathcal{N}_\mathcal{p}$_C. set $\rightarrow \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\} \subseteq \mathbb{H}^c$. whenever $\mathcal{u} \in \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\}$ and $\mathcal{v} \notin cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}$, so $cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} \subseteq \mathbb{H}$,by Proposition 3.13, thus $\mathcal{M}$ is a $T_1^{\mathcal{N}P}$_space.

Proposition 3.15.
Let $\mathfrak{f}: (\mathcal{M}, \mathfrak{I}_{\mathfrak{R}P}(\mathbb{X})) \rightarrow (\breve{\mathcal{M}}, \mathfrak{I}_{\breve{\mathfrak{R}}P}(\breve{\mathbb{X}}))$be injective $Con_{\mathcal{N}_\mathcal{p}}$ map and $\breve{\mathcal{M}}$ is a $T_0^{\mathcal{N}P}$_space, then $\mathcal{M}$ is $T_0^{\mathcal{N}P}$_space.

Proof.
Let $\mathcal{u} \neq \mathcal{v} \in \mathcal{M}$, then $\mathfrak{f}(\mathcal{u}) \neq \mathfrak{f}(\mathcal{v})$ in $\breve{\mathcal{M}}$. Since $\breve{\mathcal{M}}$is $T_0^{\mathcal{N}P}$_space, then $\exists$ $\mathcal{N}_\mathcal{p}$_O. set $\mathbb{H}$ of $\breve{\mathcal{M}}$such that $\mathfrak{f}(\mathcal{u}) \in \mathbb{H}$, $\mathfrak{f}(\mathcal{v}) \notin \mathbb{H}$. Since $\mathfrak{f}$ is $Con_{\mathcal{N}_\mathcal{p}}$, then $\mathfrak{f}^{-1}(\mathbb{H})$ is a $\mathcal{N}_\mathcal{p}$_O. set of $\mathcal{M}$, with $\mathcal{u} \in \mathfrak{f}^{-1}(\mathbb{H}), \mathcal{v} \notin \mathfrak{f}^{-1}(\mathbb{H})$.
Hence $\mathcal{M}$ is $T_0^{\mathcal{N}P}$_space.

Proposition 3.16.
Let $\breve{\mathcal{M}}$ be a $T_i^{\mathcal{N}P}$_space. If $\mathfrak{f}: (\mathcal{M}, \mathfrak{I}_{\mathfrak{R}P}(\mathbb{X})) \rightarrow (\breve{\mathcal{M}}, \mathfrak{I}_{\breve{\mathfrak{R}}P}(\breve{\mathbb{X}}))$ is bijective $Con_{\mathcal{N}_\mathcal{p}}$map, then $\mathcal{M}$is a $T_i^{\mathcal{N}P}$_space. Where $i = 1,2$.

Proof.
When *i*= 1
Let $\mathcal{u} \neq \mathcal{v} \in \mathcal{M}$, then $\mathfrak{f}(\mathcal{u}) \neq \mathfrak{f}(\mathcal{v})$ in $\breve{\mathcal{M}}$.Since , $\breve{\mathcal{M}}$ is $T_1^{\mathcal{N}P}$_space, then $\exists$ two disjoint $\mathcal{N}_\mathcal{p}$_O. sets $\mathbb{H}, \mathbb{D}$ of $\breve{\mathcal{M}}$such that $\mathfrak{f}(\mathcal{u}) \in \mathbb{H}$, $\mathfrak{f}(\mathcal{v}) \in \mathbb{D}$.
Since, $\mathfrak{f}$ is $Con_{\mathcal{N}_\mathcal{p}}$. Then $\mathfrak{f}^{-1}(\mathbb{H})$ and $\mathfrak{f}^{-1}(\mathbb{D})$ are $\mathcal{N}_\mathcal{p}$_O. sets of $\mathcal{M}$ with $\mathcal{u} \in \mathfrak{f}^{-1}(\mathbb{H})$ and $\mathcal{v} \in \mathfrak{f}^{-1}(\mathbb{D})$.
Hence $\mathcal{M}$ is $T_1^{\mathcal{N}P}$_space.
When *i*=2
Similarly, with $\mathfrak{f}^{-1}(\mathbb{H})$ and $\mathfrak{f}^{-1}(\mathbb{D})$ are distinct.

Proposition 3.17.
For every $\mathcal{u}, \mathcal{v} \in (\mathcal{M}, \mathfrak{I}_{\mathfrak{R}P}(\mathbb{X}))$, $\mathcal{u} \neq \mathcal{v} \ni cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} \neq cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\}$ there exist $\mathbb{H}, \mathbb{D}$ disjoint $\mathcal{N}_\mathcal{p}$_O. sets such that $cl_{\mathcal{N}_\mathcal{p}}(\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}) \subseteq \mathbb{H}$ and $cl_{\mathcal{N}_\mathcal{p}}(\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\}) \subseteq \mathbb{D}$ iff $(\mathcal{M}, \mathfrak{I}_{\mathfrak{R}P}(\mathbb{X}))$ is a $T_2^{\mathcal{N}P}$_space.

Proof.
Let $\mathcal{M}$ be a $T_2^{\mathcal{N}P}$_space and $\mathcal{u} \neq \mathcal{v} \in \mathcal{M}$, with $cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} \neq cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\}$, then $\exists$ disjoint $\mathcal{N}_\mathcal{p}$_O. sets $\mathbb{H}, \mathbb{D}$ such that $cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} \subseteq \mathbb{H}$ and $cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\} \subseteq \mathbb{D}$.
Since every $T_2^{\mathcal{N}P}$_space is $T_1^{\mathcal{N}P}$_space, by Lemma 3.12, we get $\mathcal{u} \in \mathcal{M}$, $cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} = \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}$. But $cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} = cl_{\mathcal{N}_\mathcal{p}}(cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}) = cl_{\mathcal{N}_\mathcal{p}}(\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\})$. Thus $cl_{\mathcal{N}_\mathcal{p}}(\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}) \subseteq \mathbb{H}$ and $cl_{\mathcal{N}_\mathcal{p}}(\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\}) \subseteq \mathbb{D}$.
Conversely, let $\mathcal{u} \neq \mathcal{v} \in \mathcal{M}$, with $cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} \neq cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\}$, then $\exists$ disjoint $\mathcal{N}_\mathcal{p}$_O. sets $\mathbb{H}, \mathbb{D}$ such that $cl_{\mathcal{N}_\mathcal{p}}(\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}) \subseteq \mathbb{H}$ and $cl_{\mathcal{N}_\mathcal{p}}(\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{v}\}) \subseteq \mathbb{D}$. Since $\{\mathcal{u}\} \subseteq \mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}$ then $cl_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\} \subseteq cl_{\mathcal{N}_\mathcal{p}}(\mathcal{K}_{\mathcal{N}_\mathcal{p}}\{\mathcal{u}\}), \forall \mathcal{u} \in \mathcal{M}$, so we get

$\mathrm{cl}_{\mathcal{N}_{\mathcal{P}}}\{u\} \subseteq \mathbb{H}$ and $\mathrm{cl}_{\mathcal{N}_{\mathcal{P}}}\{v\} \subseteq \mathbb{D}$. Then $\mathcal{M}$ be a $\mathrm{T}_2^{\mathcal{N}\mathrm{P}}$_space.

Proposition 3.18.
Let $\mathcal{M}$ be $\mathrm{T}_0^{\mathcal{N}\mathrm{P}}$_space, for every $u \neq v \in \mathcal{M}$, $v \notin \mathcal{K}_{\mathcal{N}_{\mathcal{P}}}\{u\}$ and $u \notin \mathcal{K}_{\mathcal{N}_{\mathcal{P}}}\{v\}$ such that $\mathcal{K}_{\mathcal{N}_{\mathcal{P}}}\{u\} \cap \mathcal{K}_{\mathcal{N}_{\mathcal{P}}}\{v\} = \emptyset$ iff $\mathcal{M}$ is a $\mathrm{T}_1^{\mathcal{N}\mathrm{P}}$_space.
Proof. by using Theorem 3.9, Corollary 3.10 and Theorem 2.10**.**
Proposition 3.19.
Let $\mathcal{M}$ be $\mathrm{T}_1^{\mathcal{N}\mathrm{P}}$_space, for every $u \neq v \in \mathcal{M}$ and $\mathrm{cl}_{\mathcal{N}_{\mathcal{P}}}\{u\} \neq \mathrm{cl}_{\mathcal{N}_{\mathcal{P}}}\{v\}$, $\exists\ \mathcal{N}_{\mathcal{P}}$_O. sets $\mathbb{H}$ and $\mathbb{D}$ $\ni$ $\mathrm{cl}_{\mathcal{N}_{\mathcal{P}}}(\mathcal{K}_{\mathcal{N}_{\mathcal{P}}}\{u\}) \subseteq \mathbb{H}$ and $\mathrm{cl}_{\mathcal{N}_{\mathcal{P}}}(\mathcal{K}_{\mathcal{N}_{\mathcal{P}}}\{v\}) \subseteq \mathbb{D}$ iff $\mathcal{M}$ is $\mathrm{T}_2^{\mathcal{N}\mathrm{P}}$_space.
Proof. By using Proposition 3.17. and Theorem 2.10.
Remark 3.20. The following figure. Explains the relations between spaces $\mathcal{N}\mathrm{T}_i$_spaces and $\mathrm{T}_i^{\mathcal{N}\mathrm{P}}$_spaces, where *i*=0,1,2.

$\mathcal{N}\mathrm{T}_2$_space $\rightarrow$ $\mathcal{N}\mathrm{T}_1$_space $\rightarrow$ $\mathcal{N}\mathrm{T}_0$_space

$\downarrow$ $\downarrow$ $\downarrow$

$\mathrm{T}_2^{\mathcal{N}\mathrm{P}}$_space $\rightleftarrows$ $\mathrm{T}_1^{\mathcal{N}\mathrm{P}}$_space $\rightleftarrows$ $\mathrm{T}_0^{\mathcal{N}\mathrm{P}}$_space

Fig.1
The relations between spaces $\mathcal{N}\mathrm{T}_i$_spaces and $\mathrm{T}_i^{\mathcal{N}\mathrm{P}}$_spaces, where *i*=0,1,2

4. $\mathcal{N}_{\mathcal{P}}$_Topological and Hereditary property

Definition 4.1. A map $\mathfrak{f}: (\mathcal{M}, \mathfrak{I}_{\mathfrak{R}\mathrm{P}}(\mathbb{X})) \rightarrow (\breve{\mathcal{M}}, \mathfrak{I}_{\breve{\mathfrak{R}}\mathrm{P}}(\breve{\mathbb{X}}))$ is called:
1. $\mathcal{N}_{\mathcal{P}}$_ open map ($\mathcal{N}_{\mathcal{P}}$_OM) if, $\mathfrak{f}(\mathbb{A}) \in \mathcal{N}_{\mathcal{P}}\mathrm{O}(\breve{\mathcal{M}}, \breve{\mathbb{X}})$, for each $\mathbb{A} \in \mathfrak{I}_{\mathfrak{R}\mathrm{P}}(\mathbb{X})$.
2. $\mathcal{N}_{\mathcal{P}}$_ closed map ($\mathcal{N}_{\mathcal{P}}$_CM) if, for each $\mathbb{A}^c \in \mathfrak{I}_{\mathfrak{R}\mathrm{P}}(\mathbb{X})$, $\mathfrak{f}(\mathbb{A}^c) \in \mathcal{N}_{\mathcal{P}}\mathrm{C}(\breve{\mathcal{M}}, \breve{\mathbb{X}})$.
3. $\mathcal{N}_{\mathcal{P}}$_ homeomorphism ($\mathrm{Hom}_{\mathcal{N}_{\mathcal{P}}}$) if, $\mathfrak{f}$ is bijective, $\mathrm{Con}_{\mathcal{N}_{\mathcal{P}}}$ and $\mathcal{N}_{\mathcal{P}}$_OM.

Proposition 4.2.
Let $\mathfrak{f}: (\mathcal{M}, \mathfrak{I}_{\mathfrak{R}\mathrm{P}}(\mathbb{X})) \rightarrow (\breve{\mathcal{M}}, \mathfrak{I}_{\breve{\mathfrak{R}}\mathrm{P}}(\breve{\mathbb{X}}))$ be a $\mathcal{N}_{\mathcal{P}}$_OM bijective and $\mathcal{M}$ is a $\mathrm{T}_i^{\mathcal{N}\mathrm{P}}$_space, then $\breve{\mathcal{M}}$ is $\mathrm{T}_i^{\mathcal{N}\mathrm{P}}$_space, where i = 0,1,2.
Proof.
When i= 2
Suppose that $\dot{u} \neq \dot{v} \in \breve{\mathcal{M}}$, since $\mathfrak{f}$ is injective then $\exists\ u \neq v \in \mathcal{M}$ such that $\dot{u} = \mathfrak{f}(u)$ & $\dot{v} = \mathfrak{f}(v)$. We get $\exists\ \mathbb{H}, \mathbb{D}$ are two disjoint $\mathcal{N}_{\mathcal{P}}$_O. sets in $\mathcal{M}$ such that $u \in \mathbb{H} \wedge v \in \mathbb{D}$ (because $\mathcal{M}$ is $\mathrm{T}_2^{\mathcal{N}\mathrm{P}}$_space) and $\mathbb{H} \cap \mathbb{D} = \emptyset$, since $\mathfrak{f}$ is $\mathcal{N}_{\mathcal{P}}$_OM , then $\mathfrak{f}(u), \mathfrak{f}(v)$ are $\mathcal{N}_{\mathcal{P}}$_O. sets of $\breve{\mathcal{M}}$ and $\mathfrak{f}(\mathbb{H} \cap \mathbb{D}) = \emptyset$, so $\dot{u} = \mathfrak{f}(u) \in \mathfrak{f}(\mathbb{H})$ & $\dot{v} = \mathfrak{f}(v) \in \mathfrak{f}(\mathbb{D})$. Then $\breve{\mathcal{M}}$ is an $\mathrm{T}_2^{\mathcal{N}\mathrm{P}}$_space.
Proposition 4.3. $\mathrm{T}_0^{\mathcal{N}\mathrm{P}}$_ property is $\mathcal{N}_{\mathcal{P}}$_topological property.
Proof.
suppose that $\mathfrak{f}: (\mathcal{M}, \mathfrak{I}_{\mathfrak{R}\mathrm{P}}(\mathbb{X})) \rightarrow (\breve{\mathcal{M}}, \mathfrak{I}_{\breve{\mathfrak{R}}\mathrm{P}}(\breve{\mathbb{X}}))$ is $\mathrm{Hom}_{\mathcal{N}_{\mathcal{P}}}$, $\exists\ u, v \in \mathcal{M}$ with $\breve{u} = \mathfrak{f}(u), \breve{v} = \mathfrak{f}(v)$ and $\breve{u} \neq \breve{v} \in \breve{\mathcal{M}}$, since $\mathfrak{f}$ is bijective, then $u \neq v$. Let $\mathcal{M}$ be $\mathrm{T}_0^{\mathcal{N}\mathrm{P}}$_space for u and v then $\exists\ \mathbb{H}\ \mathcal{N}_{\mathcal{P}}$_O. set $\ni u \in \mathbb{H}, v \notin \mathbb{H}$, now $\mathfrak{f}(\mathbb{H})$is $\mathcal{N}_{\mathcal{P}}$_ O. set in $\breve{\mathcal{M}}$ (because $\mathbb{H}$ is $\mathcal{N}_{\mathcal{P}}$_O. in $\mathcal{M}$ and $\mathfrak{f}$ is $\mathcal{N}_{\mathcal{P}}$_OM)), we get $\breve{u} \in \mathfrak{f}(\mathbb{H}), \breve{v} \notin \mathfrak{f}(\mathbb{H})$, hence $(\breve{\mathcal{M}}, \mathfrak{I}_{\breve{\mathfrak{R}}\mathrm{P}}(\breve{\mathbb{X}}))$ is $\mathrm{T}_0^{\mathcal{N}\mathrm{P}}$_space.
Proposition 4.4. $\mathrm{T}_1^{\mathcal{N}\mathrm{P}}$_ property is $\mathcal{N}_{\mathcal{P}}$_topological property.
Proof.
Let $\mathfrak{f}: (\mathcal{M}, \mathfrak{I}_{\mathfrak{R}\mathrm{P}}(\mathbb{X})) \rightarrow (\breve{\mathcal{M}}, \mathfrak{I}_{\breve{\mathfrak{R}}\mathrm{P}}(\breve{\mathbb{X}}))$ be $\mathrm{Hom}_{\mathcal{N}_{\mathcal{P}}}$, $\exists\ u, v \in \mathcal{M}$ $\ni \breve{u} = \mathfrak{f}(u), \breve{v} = \mathfrak{f}(v)$ and $\breve{u} \neq \breve{v} \in \breve{\mathcal{M}}$, since $\mathfrak{f}$ is bijective, then $u \neq v$. Let $\mathcal{M}$ be $\mathrm{T}_1^{\mathcal{N}\mathrm{P}}$_space for u and v then $\exists$ two $\mathcal{N}_{\mathcal{P}}$_O. sets $\mathbb{H}, \mathbb{D}$ $\ni u \in \mathbb{H}, v \notin \mathbb{H}$ and $u \notin \mathbb{D}, v \in \mathbb{D}$. Now $\mathfrak{f}(\mathbb{H}), \mathfrak{f}(\mathbb{D})$ are $\mathcal{N}_{\mathcal{P}}$_ O. sets in $\breve{\mathcal{M}}$ (because $\mathbb{H}$, $\mathbb{D}$ are $\mathcal{N}_{\mathcal{P}}$_O. sets and $\mathfrak{f}$ is $\mathcal{N}_{\mathcal{P}}$_OM , since $u \in \mathbb{H}, v \notin \mathbb{H}$ we get $\breve{u} \in \mathfrak{f}(\mathbb{H}), \breve{v} \notin \mathfrak{f}(\mathbb{H})$ and $u \notin \mathbb{D}, v \in \mathbb{D}$), we get $\breve{u} \notin \mathfrak{f}(\mathbb{D}), \breve{v} \in \mathfrak{f}(\mathbb{D})$, then $(\breve{\mathcal{M}}, \mathfrak{I}_{\breve{\mathfrak{R}}\mathrm{P}}(\breve{\mathbb{X}}))$ is $\mathrm{T}_1^{\mathcal{N}\mathrm{P}}$_space.
Proposition 4.5. $\mathrm{T}_2^{\mathcal{N}\mathrm{P}}$_ property is $\mathcal{N}_{\mathcal{P}}$_topological property.
Proof.
Let $\mathfrak{f}: (\mathcal{M}, \mathfrak{I}_{\mathfrak{R}\mathrm{P}}(\mathbb{X})) \rightarrow (\breve{\mathcal{M}}, \mathfrak{I}_{\breve{\mathfrak{R}}\mathrm{P}}(\breve{\mathbb{X}}))$ be $\mathrm{Hom}_{\mathcal{N}_{\mathcal{P}}}$, $\exists\ u, v \in \mathcal{M}$ $\ni \breve{u} = \mathfrak{f}(u), \breve{v} = \mathfrak{f}(v)$ and $\breve{u} \neq \breve{v} \in \breve{\mathcal{M}}$. Since $\mathfrak{f}$ is bijective , then $u \neq v$ and Let $\mathcal{M}$ be $\mathrm{T}_2^{\mathcal{N}\mathrm{P}}$_space for u and v then $\exists$ two $\mathcal{N}_{\mathcal{P}}$_O. sets $\mathbb{H}, \mathbb{D}$ $\ni u \in \mathbb{H}, v \notin \mathbb{H}$ and $u \notin \mathbb{D}, v \in \mathbb{D}$.
Now $\mathfrak{f}(\mathbb{H})$ and $\mathfrak{f}(\mathbb{D})$ are $\mathcal{N}_{\mathcal{P}}$_ O. sets in $\breve{\mathcal{M}}$, (because $\mathbb{H}$, $\mathbb{D}$ are $\mathcal{N}_{\mathcal{P}}$_O. sets and $\mathfrak{f}$ is $\mathcal{N}_{\mathcal{P}}$_OM and $\mathfrak{f}(\mathbb{H}) \neq \mathfrak{f}(\mathbb{D})$, such that $u \in \mathbb{H}, v \in \mathbb{D}$), we get $\mathfrak{f}(\mathbb{H}) \cap \mathfrak{f}(\mathbb{D}) = \emptyset[\mathbb{H} \cap \mathbb{D} = \emptyset$ and $\mathfrak{f}$ is one to one], however $u \in \mathbb{H} \rightarrow \breve{u} \in \mathfrak{f}(\mathbb{H})$ and $v \in \mathbb{D} \rightarrow \breve{v} \in \mathfrak{f}(\mathbb{D})$, then $(\breve{\mathcal{M}}, \mathfrak{I}_{\breve{\mathfrak{R}}\mathrm{P}}(\breve{\mathbb{X}}))$ is $\mathrm{T}_2^{\mathcal{N}\mathrm{P}}$_space.
Notes 4.6.
1- Every relative Nano_ topological space [1] is relative $\mathcal{N}_{\mathcal{P}}$_ topological space. Because, each Nano open set is $\mathcal{N}_{\mathcal{P}}$_ O. set.
2- A property δ of a $\mathcal{N}_{\mathcal{P}}$_ topological space $(\mathcal{M}, \mathfrak{I}_{\mathfrak{R}\mathrm{P}}(\mathbb{X}))$ is said $\mathcal{N}_{\mathcal{P}}$_hereditary iff $\forall$ subspace of $\mathcal{M}$ also satisfies property δ.
Proposition 4.7. $\mathrm{T}_0^{\mathcal{N}\mathrm{P}}$_ property is $\mathcal{N}_{\mathcal{P}}$_hereditary property.
Proof.
suppose that $(\mathcal{M}, \mathfrak{I}_{\mathfrak{R}\mathrm{P}}(\mathbb{X}))$ is $\mathrm{T}_0^{\mathcal{N}\mathrm{P}}$_space and $(\widehat{\mathcal{M}}, \mathfrak{I}_{\widehat{\mathfrak{R}}\mathrm{P}}(\widehat{\mathbb{X}}))$ is a subspace on $(\mathcal{M}, \mathfrak{I}_{\mathfrak{R}\mathrm{P}}(\mathbb{X}))$,
As $u, v \in \widehat{\mathcal{M}} \subseteq \mathcal{M}$ if $u \neq v \in \mathcal{M}$, we get $\exists\ \mathcal{N}_{\mathcal{P}}$_O. set $\mathbb{H}$ on $\mathcal{M}$ with $u \in \mathbb{H}, v \notin \mathbb{H}$, thus $\widehat{\mathbb{H}} = \widehat{\mathcal{M}} \cap \mathbb{H} \rightarrow \widehat{\mathbb{H}}$ is $\mathcal{N}_{\mathcal{P}}$_O. set (because $\mathbb{H}$ is $\mathcal{N}_{\mathcal{P}}$_O. set & $u \in \mathbb{H}, v \notin \mathbb{H}$) , then $u \in \widehat{\mathbb{H}}$ and $v \notin \widehat{\mathbb{H}}$, hence$(\widehat{\mathcal{M}}, \mathfrak{I}_{\widehat{\mathfrak{R}}\mathrm{P}}(\widehat{\mathbb{X}}))$ is $\mathrm{T}_0^{\mathcal{N}\mathrm{P}}$_space.
Proposition 4.7. $\mathrm{T}_i^{\mathcal{N}\mathrm{P}}$_ property is $\mathcal{N}_{\mathcal{P}}$_ hereditary property. Where $i = 1,2$

When $i = 2$

Proof.

Suppose that $(\mathcal{M}, \mathfrak{I}_{\mathfrak{RP}}(\mathbb{X}))$ is $T_2^{\mathcal{NP}}$_space and $(\widehat{\mathcal{M}}, \mathfrak{I}_{\widehat{\mathfrak{RP}}}(\widehat{\mathbb{X}}))$ is a subspace on $(\mathcal{M}, \mathfrak{I}_{\mathfrak{RP}}(\mathbb{X}))$, for $u, v \in \widehat{\mathcal{M}} \subseteq \mathcal{M}$, $u \neq v \in \mathcal{M}$, then $\exists$ two distinct $\mathcal{N}_{\mathcal{P}}$_O. sets $\mathbb{H}, \mathbb{D} \ni u \in \mathbb{H}, v \notin \mathbb{H}$ and $u \notin \mathbb{D}, v \in \mathbb{D}$, so $\widehat{\mathbb{H}} = \widehat{\mathcal{M}} \cap \mathbb{H}$ & $\widehat{\mathbb{D}} = \widehat{\mathcal{M}} \cap \mathbb{D}$.

Now $u \in \widehat{\mathbb{H}} \in \mathfrak{I}_{\widehat{\mathfrak{RP}}}(\widehat{\mathbb{X}})$ and $v \in \widehat{\mathbb{D}} \in \mathfrak{I}_{\widehat{\mathfrak{RP}}}(\widehat{\mathbb{X}})$ (because $u \in \mathbb{H} \in \mathfrak{I}_{\mathfrak{RP}}(\mathbb{X})$ and $v \in \mathbb{D} \in \mathfrak{I}_{\mathfrak{RP}}(\mathbb{X})$) and since $\mathbb{H} \cap \mathbb{D} = \emptyset$, hence $\widehat{\mathbb{H}} \cap \widehat{\mathbb{D}} = (\widehat{\mathcal{M}} \cap \mathbb{H}) \cap (\widehat{\mathcal{M}} \cap \mathbb{D}) = \widehat{\mathcal{M}} \cap (\mathbb{H} \cap \mathbb{D}) = \widehat{\mathcal{M}} \cap \emptyset = \emptyset$. Then $(\widehat{\mathcal{M}}, \mathfrak{I}_{\widehat{\mathfrak{RP}}}(\widehat{\mathbb{X}}))$ is $T_2^{\mathcal{NP}}$_space.

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تطبيقات بديهيات الفصل للنانو الخماسي ($\mathcal{N}_{\mathcal{P}}$) من خلال المجموعات المفتوحة – $\mathcal{N}_{\mathcal{P}}$

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الملخص

الهدف الرئيسي لهذا البحث هو استخدام مفهوم المجموعات المفتوحة نانو خماسي ($\mathcal{N}_{\mathcal{P}}$ _ open sets). لتقديم فئات جديدة من بديهيات الفصل في الفضاء التبولوجي نانو خماسي، هذه الفئات الجديدة هي $T_i^{\mathcal{NP}}$_space , $i = 0,1,2$ وتم دراسة بعض الخصائص الأساسية لهذه الفضاءات، ناقشنا أيضا العلاقات بينها وبين بديهيات الفصل نانو $\mathcal{N}T_i$_spaces, $i = 0,1,2$، وكذلك تناول البحث العلاقة بين بديهيات الفصل عبر مجموعة النواة المرتبطة بالمجموعة المغلقة التي استخُدمت لإثبات بعض النظريات المتعلقة بها ، كما تمت مناقشة الخصائص الوراثية والتبولوجية.