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Using Touchard Polynomials Method for Solving Volterra-Fredholm Integro-Differential Equations

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ABSTRACT

The goal of this paper is to introduce numerical solution for Volterra-Fredholm integro-differential equations of the second kind. The proposed method is Touchard polynomials method, and this technique transforms the integro-differential equations to the system of algebraic equations. Four examples are presented in order to illustrate the accuracy and efficiency of this method.

1. Introduction

Much attention has been given to study the integral and integro-differential equations because of their applications in different ways such as fluid dynamics, biological models, engineering and physical models. The integral equation is the equation that has no derivative. Several studies have been considered such type of equations by using different numerical methods, for example, the author in [1] used Bernstein polynomials method for solving Volterra-Fredholm integral equations of the second kind. While, the same type of equations has been solved by using the Boubaker polynomials method [2]. Furthermore, the Touchard polynomials method has been applied to the same type of equations in [3]. A paper by [4] used Lagrange polynomials method for solving Volterra-Fredholm integral equations. The Touchard polynomials method has been applied in [5] for solving linear and nonlinear Volterra (Fredholm) integral equations. Then, several examples are given to illustrate the proposed method. In recent years, Volterra-Fredholm integral equation was solved by using Hosoya Polynomials [6]. Moreover, the hyperbolic basis functions has been used to solve the second kind linear Volterra-Fredholm integral equation [7]. On the other hand, the integro-differential equations played important role in various fields and they have been taken much interesting by many studies such as in [8] considered Bernstein polynomials method for solving Volterra-Fredholm integro-differential equations of the second kind. Beside that, the Reliable Iterative method has been used to solve the same type of equations [9]. The Modified Decomposition method (MDM) was applied to Volterra-Fredholm integro-differential equation using Maple [10]. In addition, Block Pulse Functions and their operational matrices are used to solve Volterra-Fredholm integro-differential equation [11]. Lagrange polynomials method, Repeated Trapezoidal method, and Repeated Simpson's 1/3 have been applied in [12-13] for solving such types of equations. Finally, the Power series, Chebyshev polynomials, and Legendre's polynomials method are considered in [14] for solving the same equations. In the present paper, the Touchard polynomials method has been considered to solve the linear Volterra-Fredholm integro-differential equations of the second kind.

The present paper is organized as follows: the proposed method is presented in section two. The solution of such types of equations has been described in section three. Some numerical examples are given in section four. Then, the comparison with other results has been presented in section five. Finally, the conclusions of this paper are stated in section six.

Generally, the Volterra-Fredholm integro-differential equation of the second kind is given in the form:

$$\sum_{g=0}^{t}\delta_g(\omega)F^g(\omega)=\phi(\omega)+\beta_1\int_a^{\omega}u_1(\omega,h)F(h)dh+\beta_2\int_a^{b}u_2(\omega,h)F(h)dh$$

(1,1)

where the initial condition $F^g(a)=F_g, g=1,2,...,t$, for each $a,b,\beta_1,\beta_2\in R$, $\phi(\omega),u_1(\omega,h),u_2(\omega,h)$, $\delta_g(\omega), g=1,2,...,t$ and $\delta_g(\omega)\neq 0$ are known functions which have derivative on the interval $[a,b]$ and $F(\omega)$ is the unknown function that will be determined.

2. Touchard Polynomials Method

The Touchard polynomials has been studied since 1939 by Jacques Touchard which is a French mathematician. Touchard polynomials is defined as [5]:

$$\Gamma_\tau(r)=\sum_{\rho=0}^{\tau}\binom{\tau}{\rho}r^\rho \quad (2,1)$$

The first six terms of Touchard polynomials are:

$\Gamma_0(r)=1$

$\Gamma_1(r)=1+r$

$\Gamma_2(r)=1+2r+r^2$

$\Gamma_3(r)=1+3r+3r^2+r^3$

$\Gamma_4(r)=1+4r+6r^2+4r^3+r^4$

$\Gamma_5(r)=1+5r+10r^2+10r^3+5r^4+r^5.$

2.1 The Matrix Formulation for (T-Ps)

In this section, the matrix formulation of the proposed method has been presented. The Touchard polynomial can be written as a linear combination of a Touchard basis functions in terms of dot scalar as:

$$\Gamma_\tau(r)=\xi_0\Gamma_0(r)+\xi_1\Gamma_1(r)+\xi_2\Gamma_2(r)+\cdots+\xi_\tau\Gamma_\tau(r)=\sum_{q=0}^{\tau}\xi_q\Gamma_q(r)$$

(2,1)

where $\xi_q, q=0,1,2,...,\tau$ are the unknown cofficients that will be found. Equation (2,1) can be written as a dot scalar of two vectors:

$$\Gamma_\tau(r)=\begin{bmatrix}\Gamma_0(r) & \Gamma_1(r) & \Gamma_2(r) & \cdots & \Gamma_\tau(r)\end{bmatrix}\cdot\begin{bmatrix}\xi_0\\ \xi_1\\ \xi_2\\ \vdots\\ \xi_\tau\end{bmatrix}.$$

(2,2)

Equation (2,2) can be convert to the form:

$$\Gamma_\tau(r)=\begin{bmatrix}1 & r & r^2 & \cdots & r^\tau\end{bmatrix}\cdot\begin{bmatrix}\kappa_{00} & \kappa_{01} & \kappa_{02} & \cdots & \kappa_{0\tau}\\ 0 & \kappa_{11} & \kappa_{12} & \cdots & \kappa_{1\tau}\\ 0 & 0 & \kappa_{22} & \cdots & \kappa_{2\tau}\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & 0 & \kappa_{\tau\tau}\end{bmatrix}\cdot\begin{bmatrix}\xi_0\\ \xi_1\\ \xi_2\\ \vdots\\ \xi_\tau\end{bmatrix}$$

(2,3)

where κ's are the cofficients of the power basis that are used to detremine the respective Touchard polynomial.

3. Solution for Volterra-Fredholm Integro-Differential Equations of the Second Kind

This section presents an approximate solution of Volterra-Fredholm integro-differential equations by using Touchard polynomials method.

Consider the VFIDE2K which is given in equation (1,1)

$$\sum_{g=0}^{t}\delta_g(\omega)F^g(\omega)=\phi(\omega)+\beta_1\int_a^{\omega}u_1(\omega,h)F(h)dh+\beta_2\int_a^{b}u_2(\omega,h)F(h)dh$$

(3,1)

Let $F(\omega)=\Gamma_\tau(r)$, then

$$F(\omega)=\xi_0\Gamma_0(r)+\xi_1\Gamma_1(r)+\xi_2\Gamma_2(r)+\cdots+\xi_\tau\Gamma_\tau(r) \quad (3,2)$$

where $\Gamma_\tau(r)$ is the Touchard polynomial which was defined in equation (1,1) and $\xi_0,\xi_1,\xi_2,\cdots,\xi_\tau$ are the unknown coefficients that will be determined.

Equation (3,2) can be written as a dot product:

$$F(\omega)=\begin{bmatrix}\Gamma_0(r) & \Gamma_1(r) & \Gamma_2(r) & \cdots & \Gamma_\tau(r)\end{bmatrix}\begin{bmatrix}\xi_0\\ \xi_1\\ \xi_2\\ \vdots\\ \xi_\tau\end{bmatrix}.$$

(3,3)

Equation (3,3) can be converted to be:

$$F(\omega)=\begin{bmatrix}1 & r & r^2 & \cdots & r^\tau\end{bmatrix}\begin{bmatrix}\kappa_{00} & \kappa_{01} & \kappa_{02} & \cdots & \kappa_{0\tau}\\ 0 & \kappa_{11} & \kappa_{12} & \cdots & \kappa_{1\tau}\\ 0 & 0 & \kappa_{22} & \cdots & \kappa_{2\tau}\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & \kappa_{\tau\tau}\end{bmatrix}\begin{bmatrix}\xi_0\\ \xi_1\\ \xi_2\\ \vdots\\ \xi_\tau\end{bmatrix}.$$

(3,4)

Now, substituting equation (3,3) into equation (3,1) we get:

$$\sum_{g=0}^{t}\delta_g(\omega)\left[\begin{bmatrix}\Gamma_0(r) & \Gamma_1(r) & \Gamma_2(r) & \cdots & \Gamma_\tau(r)\end{bmatrix}\begin{bmatrix}\xi_0\\ \xi_1\\ \xi_2\\ \vdots\\ \xi_\tau\end{bmatrix}\right]^g=\phi(\omega)$$

$$+\beta_1\int_a^{\omega}u_1(\omega,h)\begin{bmatrix}\Gamma_0(r) & \Gamma_1(r) & \Gamma_2(r) & \cdots & \Gamma_\tau(r)\end{bmatrix}\begin{bmatrix}\xi_0\\ \xi_1\\ \xi_2\\ \vdots\\ \xi_\tau\end{bmatrix}dh$$

$$+\beta_2\int_a^{b}u_2(\omega,h)\begin{bmatrix}\Gamma_0(r) & \Gamma_1(r) & \Gamma_2(r) & \cdots & \Gamma_\tau(r)\end{bmatrix}\begin{bmatrix}\xi_0\\ \xi_1\\ \xi_2\\ \vdots\\ \xi_\tau\end{bmatrix}dh.$$

(3,5)

Next, applying equation (3,4) into equation (3,5) we get:

$$\sum_{g=0}^{t}\delta_g\left[\begin{bmatrix}\Gamma_0(r) & \Gamma_1(r) & \Gamma_2(r) & \cdots & \Gamma_\tau(r)\end{bmatrix}\begin{bmatrix}\xi_0\\ \xi_1\\ \xi_2\\ \vdots\\ \xi_\tau\end{bmatrix}\right]^g=\phi(\omega)$$

$$+\beta_1\int_a^{\omega} u_1(\omega,h)\begin{bmatrix}1 & r & r^2 & \cdots & r^{\tau}\end{bmatrix}\begin{bmatrix}\kappa_{00} & \kappa_{01} & \kappa_{02} & \cdots & \kappa_{0\tau}\\ 0 & \kappa_{11} & \kappa_{12} & \cdots & \kappa_{1\tau}\\ 0 & 0 & \kappa_{22} & \cdots & \kappa_{2\tau}\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & \kappa_{\tau\tau}\end{bmatrix}\begin{bmatrix}\xi_0\\ \xi_1\\ \xi_2\\ \vdots\\ \xi_{\tau}\end{bmatrix}dh$$

$$+\beta_2\int_a^{b} u_2(\omega,h)\begin{bmatrix}1 & r & r^2 & \cdots & r^{\tau}\end{bmatrix}\begin{bmatrix}\kappa_{00} & \kappa_{01} & \kappa_{02} & \cdots & \kappa_{0\tau}\\ 0 & \kappa_{11} & \kappa_{12} & \cdots & \kappa_{1\tau}\\ 0 & 0 & \kappa_{22} & \cdots & \kappa_{2\tau}\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & \kappa_{\tau\tau}\end{bmatrix}\begin{bmatrix}\xi_0\\ \xi_1\\ \xi_2\\ \vdots\\ \xi_{\tau}\end{bmatrix}dt$$

(3,6)

Now, computing the integrations on the right side of equation (3,6), and this equation will simplified as a linear equation included ω as a variable. Then, choosing ω_j in the interval $[a,b]$ by the formula $\omega_j = a + jd$ where $d = \frac{b-a}{\tau}, j = 0,1,2,\cdots,\tau.$ After that, a system of a linear equations consisting of $\tau+1$ unknown coefficients $\xi_0,\xi_1,\xi_2,\cdots,\xi_{\tau}$ can be solved by using Gauss Elimination to determine the values of these unknown coefficients. These steps are summarized and presented in the following Algorithm.

(Algorithm of Solution)

Input: $(F(\omega),u(\omega,h),\phi(\omega),\omega,a,b,\beta_1,\beta_2)$

Output: The polynomials of the degree τ.

Step 1: Select the degree of the Touchard Polynomial method.

Step 2: Put the Touchard polynomials in the unknown function of the VFIDE2K.

Step 3: Compute the following

1- Volterra Integral

2- Fredholm Integral

3- The term $\sum_{g=0}^{t}\delta_g(\omega)F^{g}(\omega)$.

Step 4: Compute $\xi_0,\xi_1,\xi_2,\cdots,\xi_{\tau}$.

End.

4. Numerical Examples

In this section, four numerical examples are given in order to illustrate the proposed method. The computations associated with the examples have been performed by using Matlab.

Example 1: Consider the following VFIDE2K [8]

$F'(\omega) = 2e^{\omega} - 2 + \int_0^{\omega} F(h)dh + \int_0^{1} F(h)dh,$ with the initial condition $F(0)=0$ and the exact solution is $F(\omega)=\omega e^{\omega}$.

Table 1: Numerical Results for Example 1.

ω	Exact Solution	Yapp n=2	Yapp n=3	Yapp n=4	L.S.E
0.0	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.1105	0.1976	0.1059	0.1106	0.0000
0.2	0.2443	0.4467	0.2322	0.2445	0.0000
0.3	0.4050	0.7474	0.3848	0.4051	0.0000
0.4	0.5967	1.0997	0.5698	0.5964	0.0000
0.5	0.8244	1.5035	0.7929	0.8234	0.0000
0.6	1.0933	1.9588	1.0601	1.0916	0.0000
0.7	1.4096	2.4657	1.3774	1.4075	0.0000
0.8	1.7804	3.0241	1.7507	1.7781	0.0000
0.9	2.2136	3.6341	2.1859	2.2115	0.0000
1.0	2.7183	4.2957	2.6890	2.7162	0.0000

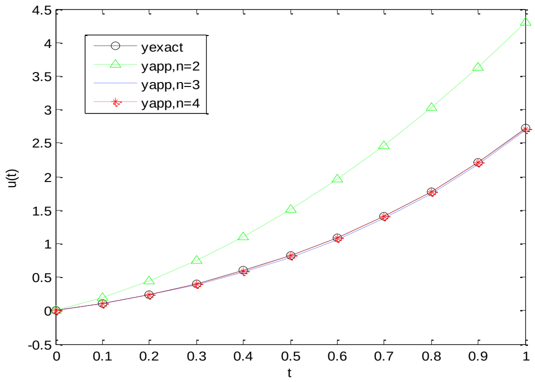

Fig. 1: Exact and Approximate Solutions for Example 1.

Example 2: Consider the following VFIDE2K [15]

$F'''(\omega) = 2\sin(\omega) - \omega - 3 + \int_0^{\omega}(\omega-h)F(h)dh + \int_0^{\frac{\pi}{2}} F(h)dh,$ with the initial conditions $F(0)=F'(0)=1$ and $F''(0)=-1$. The exact solution is $F(\omega)=\sin(\omega)+\cos(\omega)$.

Table 2: Numerical Results for Example 2.

ω	Exact Solution	Yapp n=4	Yapp n=5	L.S.E
0.0	1.0000	1.0000	1.0000	0.0000
0.1	1.0948	1.0948	1.0948	0.0000
0.2	1.1787	1.1788	1.1788	0.0000
0.3	1.2509	1.2513	1.2509	0.0000
0.4	1.3105	1.3117	1.3107	0.0000
0.5	1.3570	1.3596	1.3575	0.0002
0.6	1.3900	1.3949	1.3909	0.0008
0.7	1.4091	1.4176	1.4107	0.0027
0.8	1.4141	1.4277	1.4168	0.0078
0.9	1.4049	1.4257	1.4095	0.0209
1.0	1.3818	1.4119	1.3890	0.0524

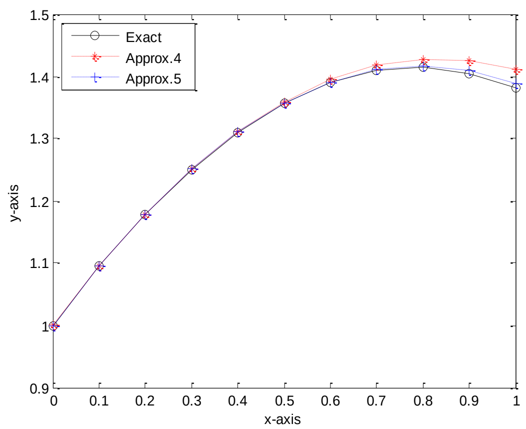

Fig. 2: Exact and Approximate Solutions for Example 2.

Example 3: Consider the following VFIDE2K [9] $F'(\omega)=1+\int_0^{\omega}(\omega-h)F(h)dh+\int_0^1 \omega hF(h)dh,$ with the initial condition $F(0)=1$ and the exact solution is $F(\omega)=e^{\omega}$.

Table 3: Numerical Results for Example 3.

ω	Exact Solution	Yapp n=7	Yapp n=8	L.S.E
0.0	1.0000	1.0000	1.0000	0.0000
0.1	1.1052	1.1053	1.1053	0.0000
0.2	1.2214	1.2222	1.2220	0.0000
0.3	1.3499	1.3524	1.3519	0.0000
0.4	1.4918	1.4983	1.4970	0.0000
0.5	1.6487	1.6623	1.6596	0.0001
0.6	1.8221	1.8475	1.8417	0.0004
0.7	2.0138	2.0566	2.0444	0.0009
0.8	2.2255	2.2922	2.2674	0.0018
0.9	2.4596	2.5560	2.5084	0.0024
1.0	2.7183	2.8488	2.7617	0.0019

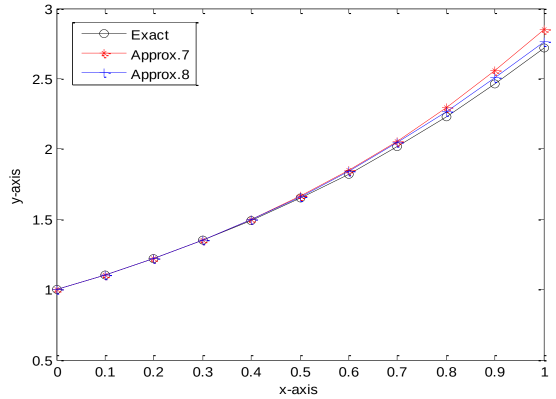

Fig. 3: Exact and Approximate Solutions for Example 3.

Example 4: Consider the following VFIDE2K [14] $F'''(\omega)=\frac{\omega^2}{2}+\int_0^{\omega}F(h)dh+\int_{-\pi}^{\pi}\omega F(h)dh,$ with the initial conditions $F(0)=F'(0)=1$ and $-F''(0)=1.$ The exact solution is $F(\omega)=\omega+\cos(\omega).$

Table 4: Numerical Results for Example 4.

ω	Exact Solution	Yapp n=4	Yapp n=5	L.S.E
0.0	1.0000	1.0000	1.0000	0.0000
0.1	1.0950	1.0950	1.0950	0.0000
0.2	1.1801	1.1796	1.1800	0.0000
0.3	1.2553	1.2538	1.2552	0.0000
0.4	1.3211	1.3174	1.3207	0.0000
0.5	1.3776	1.3702	1.3770	0.0000
0.6	1.4253	1.4121	1.4243	0.0000
0.7	1.4648	1.4432	1.4635	0.0000
0.8	1.4967	1.4634	1.4951	0.0000
0.9	1.5216	1.4728	1.5203	0.0000
1.0	1.5403	1.4716	1.5401	0.0000

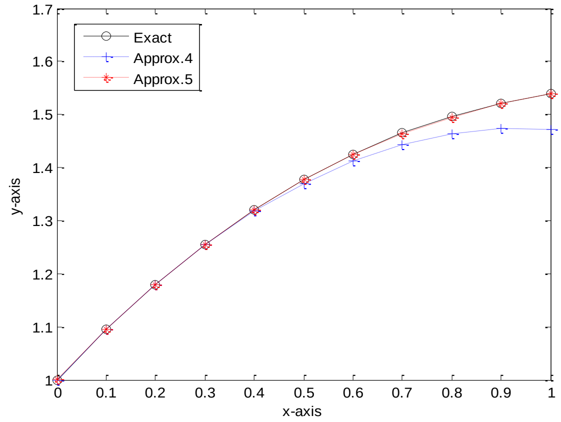

Fig. 4: Exact and Approximate Solutions for Example 4.

5. Comparison With Other Methods

In this section, the comparison of our results with some existing results that have been computed by using Bernstein polynomials method are presented. The comparison had been done for example one and displayed that the results are almost the same between the two numerical methods even with their degrees are different. But for this example, the proposed method had reached up to the exact solution faster than Bernstein polynomials method. Therefore, Touchard polynomials method is accurate and more efficient than Bernstein polynomials method of Example 1.

Table 5: Comparison Numerical Results for Example 1.

ω	Exact Solution	Touchard Polynomials Yapp, n=4	Bernestein Polynomials Yapp, n=5 [8]
0.0	0.0000	0.0000	0.0000
0.1	0.1105	0.1106	0.1105
0.2	0.2443	0.2445	0.2443
0.3	0.4050	0.4051	0.4050
0.4	0.5967	0.5964	0.5968
0.5	0.8244	0.8234	0.8244
0.6	1.0933	1.0916	1.0933
0.7	1.4096	1.4075	1.4097
0.8	1.7804	1.7781	1.7805
0.9	2.2136	2.2115	2.2137
1.0	2.7183	2.7162	2.7184

6. Conclusion

The types of equations are difficult to solve analytically, so they required approximate solutions, and for this purpose, Touchard polynomials method has been applied to introduce the numerical solution for these equations. The obtained results showed that this method is efficient and accurate to estimate the solution of these equations. Also, we noticed that when the degree of the proposed method is increased, the error is decreased as a result of that. It is clear that from the Tables and Figures, more accurate results can be obtained by increasing the degree of the proposed method. Furthermore, the results of this study are compared with some existing results that have computed by different method. The comparison showed that the proposed method is an accurate and efficient for solving such types of equations. A suggestion for future work may include the solution algorithm can be programmed and then executed programmatically. Then, can apply the same method for solving the nonlinear Volterra-Fredholm integro-differential equations of the second kind.

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طريقة متعددات حدود توجارد لحل معادلة فولتيرا– فريدهولم التفاضلية التكاملية الخطية

محمد خالد شاحوذ

مديرية تربية الانبار ، الرمادي ، العراق

الملخص

الهدف من هذا البحث هو تقديم حل عددي لمعادلة فولتيرا – فريدهولم التفاضلية التكاملية الخطية من النوع الثاني باستخدام متعددات حدود توجارد. الخوارزمية والامثلة المعطاة هي لتوضيح الحل باستخدام هذه الطريقة ومقارنته مع الحل الدقيق. اضف الى ذلك، قد تمت مقارنة النتائج مع طريقة متعددات حدود برنشتاين والمقارنة كانت ممتازة ومتوافقة بين الحلول الدقيقة والحلول العددية. اثبتت المقارنة ان الطريقة المقترحة وصلت الى الحل الدقيق في درجة اقل من متعددات حدود برنشتاين، مما يبين ان طريقة متعددات حدود توجارد هي طريقة جيدة وفعالة لحل هذا النوع من المعادلات.