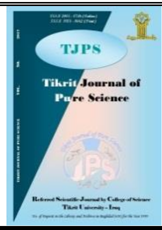
*Tikrit Journal of Pure Science Vol. 26 (5) 2021* TJPS

Tikrit Journal of Pure Science

*ISSN: 1813 – 1662 (Print) --- E-ISSN: 2415 – 1726 (Online)*

**Journal Homepage:** http://tjps.tu.edu.iq/index.php/j



# A New Method for Solving Quadratic Fractional Programing Problems

**Mediya B. Mrakhan, Basiya K.Abdulrahim, Rzgar F. Mahmood, Aram Bajalan, Shwan Adnan Ali**

*Department of Mathematics, College of Education, University of Garmian, Kurdistan Region, Iraq*

https://doi.org/10.25130/tjps.v26i5.182

**ARTICLE INFO.**

**Article history:**

**-Received:** 13 / 12 / 2020

**-Received in revised form:** 28 / 2 / 2021

**-Accepted:** 19 / 7 / 2021

**-Final Proofreading:** 7 / 11 / 2021

**-Available online:** 21/ 11 / 2021

**Keywords:** Linear Programming Problem, Fractional Programming Problem, QFPP, RBM-Technique.

**Corresponding Author:**

**Name:** Mediya B. Mrakhan

**E-mail:**

media.bawaxan@Garmian.edu.krd

**Tel:**

## ABSTRACT

In this article, the given algorithms were expanded and a methodology was developed to solve an objective function of a quadratic fractional programming problem (QFPP) with linear constraints. A new method called RBM was introduced to directly solve the problem with optimal solution. Finally, the nonlinear problem was transformed to a linear programming problem with two constraints. No more conversion was made to the initial problem. Numerical examples were illustrated to show the efficiency of the method. In addition, the computer operation of the current algorithms was discussed by using MATLAB 2013a to solve a constructed numerical example.

## 1. Introduction

Since they have been used in the production planning, financial and corporate organizing, medical services and hospital planning, and other fields, QFPP with linear maximums issues (i.e., ratio of goals with numerator and denominator) have generated a lot more research and interest. Many techniques to solve such issues have been introduced in [4]. Abdulrahim uses feasible directions to solve QFPP [2]. The method, known as the modified objective function, is used to resolve this linear fractional program by optimizing a series of linear programs and only recalculating the objective function's local gradient. Moreover, many aspects of linear fraction program duality and sensitivity analysis are mentioned in [6]. Fukushima and Hayashi solve QFPPs with quadratic constraints [5]. Abdulrahim uses the feasible path creation and updated simplex method to solve QFPP [1]. In this article, RBM-technique is presented for solving problems with quadratic fractional goals and inequality constraints. Then, a detailed overview of the issue as well as the key findings are discussed. Moreover, the steps of the current new algorithm with two numerical examples are presented to clarify how this new algorithm works. Finally, the key conclusion of this solution procedure are provided as well.

## 2. Quadratic Fractional Programming Problem

**Definition 2.1: A Quadratic Programming Problem [1-3]**

The quadratic programming problem formulation has the following structure:

$$\text{Maximize (Minimize)} Z = (\alpha + C^T X + \tfrac{1}{2} X^T G X)$$

Conditional on:

$$AX \begin{bmatrix} \leq \\ \geq \\ = \end{bmatrix} B$$

$$X \geq 0$$

Where $A = (a_{ij})_{m\times n}$ indicates matrix of coefficients $\forall i = 1,2,\dots,m,\ \ j = 1,2,..,n,\ B = (b_1, b_2, \dots, b_m)^T$, $X = (x_1, x_2, \dots, x_n)^T$, $C^T = (c_1\,, c_2\,, \dots, c_n)^T$, and $G = (g_{ii})_{n\times n}$ is a symmetric square matrix that is positive definite or positive semi-definite. If $T$ is transposed and $\alpha$ is scalar, the requirements will be linear, and the optimal solution will be quadratic.

**Definition 2.2: Fractional Programming Problem with Quadratic Fractions [1]**

QFPP's mathematical programming issue can be expressed as follows:

Maximize (Minimize) $Z = \frac{(\alpha_1 + C_1^T X + \frac{1}{2} X^T G_1 X)}{(\alpha_2 + C_2^T X + \frac{1}{2} X^T G_2 X)}$

Subject to :

$$AX \begin{bmatrix} \leq \\ \geq \\ = \end{bmatrix} X$$

$$X \geq 0$$

Where $G_1, G_2$ are $(n \times n)$ coefficients matrix of $G_1, G_2$ matrices that are symmetric. Except transposed (T), where X is an n-dimensional vector of decision variables, all vectors are considered columns, $C_1, C_2$ are the n-dimensional vectors of the constants, $\alpha_1, \alpha_2$ are scalars and $B$ is the m-dimensional vector of the constants. The problem to be resolved in this study has the following general structure:

Maximize $Z = \frac{(c^T X + \gamma)(e^T X + \delta)}{(d^T x + \beta)(f^T x + \varepsilon)} = \left(\frac{(c^T X + \gamma)}{(d^T x + \beta)}\right)\left(\frac{(e^T X + \delta)}{(f^T x + \varepsilon)}\right) = \frac{Z_1}{Z_2}$

or Maximize $Z = \frac{(c^T X + \gamma)(e^T X + \delta)}{(d^T x + \beta)(f^T x + \varepsilon)} = \left(\frac{(c^T X + \gamma)}{(f^T X + \varepsilon)}\right)\left(\frac{(e^T X + \delta)}{(d^T X + \beta)}\right) = \frac{Z_1}{Z_2}$

Subject to:

$$AX \begin{bmatrix} \leq \\ \geq \\ = \end{bmatrix} B$$

$$X \geq 0$$

Where $X \in R^n$, $A$ is an $m \times n$ matrix; $c, e, d$ and $f$ are $n$ −vectors; $B \in R^m$ and $\gamma, \beta, \delta, \varepsilon$ are scalar constants. Moreover $f^T X + \varepsilon$, $d^T X + \beta > 0$ everywhere in $X$.

## 3. RBM-Technique and Formulation

The RBM-Technique is a systematic algorithm that involves getting from one simple feasible solution (on vertex) to the other in a specified order for increasing the value of optimal solution. If the objective function increases with each jump, no foundation will be replicated; so, there is no need to return to a previously covered vertex. The process should lead to the optimized vertex in a finite number of steps because the number of vertices is limited.

For resolving quadratic programming issues, the RBM-Technique is considered an optimization (step-by-step) method. It consists of the following procedures:

1) Having a trail basic feasible solution to constraint equations.
2) Testing whether there is an optimal solution.
3) Improving the first trial solution by a set of rules.
4) Repeating the process until an optimal solution is obtained.

$\Delta_{ji} = c_{ji} - c_{Bi}^T x_{ij}, i = 1,2,3,4\ j = 1,2, \ldots, m + n$

$\Delta_{ja} = z_2 \Delta_{j1} - z_1 \Delta_{j2},\ \ \Delta_{jb} = z_4 \Delta_{j3} - z_3 \Delta_{j4}, j = 1,2, \ldots, m + n$

$z_1 = c_{B1} v_B + \gamma, z_2 = c_{B2} v_B + \delta, z_3 = c_{B3} v_B + \epsilon, z_4 = c_{B4} v_B + \beta$

$Z = \frac{z_1 z_2}{z_3 z_4} = \frac{Z_1}{Z_2}$

$\Delta_j = \Delta_{ja} \times \Delta_{jb}$

Here, $c_{ji}, i = 1,2,3,4, j = 1,2, \ldots, m + n$ are the coefficients of the objective function's essential and non-basic variables and $c_{Bi}, i = 1,2,3,4$ are the simple variables' coefficients in the optimization problem.

As for the RBM-Technique algorithm for solving QFPP, it can be categorized as follows:

**Step1:** Write the basic way of the issue; attach slack and artificial variables to a limitation; and use RBM-Technique to begin.

**Step2:** Create an estimate $\Delta_j$ by using the formula $\Delta_j = \Delta_{ja} \times \Delta_{jb}$, and then enter it in the RBM-Technique table at the start.

**Step3:** Use the RBM-Technique method to find the solution.

**Step4:** In step3, verify the solution for feasibility; if it is achieved, then go to step 5; if not, then use the dual RBM-Technique to eliminate the infeasibility.

**Step5:** Verify that the solution is optimal; if all $\Delta_j \leq 0$, then the solution will be perfect; if not, then return to step 3.

## 4. Numerical Examples

**Example 4.1:**

The following QFPP is taken into consideration:

$Max. Z = \frac{(4x_1 + 2x_2)(2x_1 + x_2 + 3)}{(x_1 + x_2 + 2)(2x_1 + x_2 + 5)} = \frac{4x_1 + 2x}{2x_1 + x_2 + 5} \times \frac{2x_1 + x_2 + 3}{x_1 + x_2 + 2}$

Subject to:

$$x_1 + x_2 \leq 6$$
$$8x_1 + 4x_2 \leq 32$$
$$x_1 \leq 3$$
$$x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

**Solution:**

By solving example 4.1 via the RBM-Technique and applying the algorithm in section 4, the RBM-Technique's initial table was reached, as shown in table 1.1. After one iteration, the RBM-Technique's final table was achieved, as clarified in table 1.2.

**Table 1.1: RBM-Technique's initial table for Example 4.1**

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $c_{j1}$ | 4 | 2 | 0 | 0 | 0 | 0 | |
| | | | | | $c_{j2}$ | 1 | 1 | 0 | 0 | 0 | 0 | |
| | | | | | $c_{j3}$ | 2 | 1 | 0 | 0 | 0 | 0 | |
| | | | | | $c_{j4}$ | 2 | 1 | 0 | 0 | 0 | 0 | |
| B.V. | $c_{B1}$ | $c_{B2}$ | $c_{B3}$ | $c_{B4}$ | $V_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | Main-Ratio |
| $x_3$ | 0 | 0 | 0 | 0 | 3 | 0 | 1 | 1 | 0 | 0 | 0 | 6 |
| $x_4$ | 0 | 0 | 0 | 0 | 8 | 0 | 4 | 0 | 1 | 0 | 0 | 4 |
| $x_5$ | 0 | 0 | 0 | 0 | 3 | 1 | 0 | 0 | 0 | 1 | 0 | 3 ⬅ |
| $x_6$ | 0 | 0 | 0 | 0 | 5 | 0 | 1 | 0 | 0 | 0 | 1 | – |
| $z_1 = 0$ | | | | | $\Delta_{j1}$ | $-4$ | $-2$ | 0 | 0 | 0 | 0 | |
| $z_2 = 5$ | | | | | $\Delta_{j2}$ | $-2$ | $-1$ | 0 | 0 | 0 | 0 | |
| $Z_1 = z_1/z_2 = 0$ | | | | | $\Delta_{ja}$ | $-20$ | $-10$ | 0 | 0 | 0 | 0 | |
| $z_3 = 3$ | | | | | $\Delta_{j3}$ | $-2$ | $-1$ | 0 | 0 | 0 | 0 | |
| $z_4 = 2$ | | | | | $\Delta_{j4}$ | $-1$ | $-1$ | 0 | 0 | 0 | 0 | |
| $Z_2 = z_3/z_4 = 3/2$ | | | | | $\Delta_{jb}$ | $-1$ | 1 | 0 | 0 | 0 | 0 | |
| $Z = Z_1/Z_2$ | | | | | $\Delta_j$ | (20) ⬆ | $-10$ | 0 | 0 | 0 | 0 | |

**Table 1.2: RBM-Technique's final table for Example 4. 1**

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $c_{j1}$ | 4 | 2 | 0 | 0 | 0 | 0 |
| | | | | | $c_{j2}$ | 1 | 1 | 0 | 0 | 0 | 0 |
| | | | | | $c_{j3}$ | 2 | 1 | 0 | 0 | 0 | 0 |
| | | | | | $c_{j4}$ | 2 | 1 | 0 | 0 | 0 | 0 |
| B.V. | $c_{B1}$ | $c_{B2}$ | $c_{B3}$ | $c_{B4}$ | $V_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
| $x_3$ | 0 | 0 | 0 | 0 | 3 | 1 | 0 | 1 | 0 | 0 | $-1$ |
| $x_4$ | 0 | 0 | 0 | 0 | 8 | 8 | 0 | 0 | 1 | 0 | $-4$ |
| $x_2$ | 0 | 0 | 0 | 0 | 3 | 1 | 0 | 0 | 0 | 1 | 0 |
| $x_6$ | 2 | 1 | 1 | 1 | 5 | 0 | 1 | 0 | 0 | 0 | 1 |
| $z_1 = 10$ | | | | | $\Delta_{j1}$ | $-4$ | 0 | 0 | 0 | 0 | 2 |
| $z_2 = 10$ | | | | | $\Delta_{j2}$ | $-1$ | 0 | 0 | 0 | 0 | 1 |
| $Z_1 = z_1/z_2 = 1$ | | | | | $\Delta_{ja}$ | $-30$ | 0 | 0 | 0 | 0 | 10 |
| $z_3 = 8$ | | | | | $\Delta_{j3}$ | $-2$ | 0 | 0 | 0 | 0 | 1 |
| $z_4 = 7$ | | | | | $\Delta_{j4}$ | $-2$ | 0 | 0 | 0 | 0 | 1 |
| $Z_2 = z_3/z_4 = 8/7$ | | | | | $\Delta_{jb}$ | 2 | 0 | 0 | 0 | 0 | $-1$ |
| | | | | | $\Delta_j$ | $-60$ | 0 | 0 | 0 | 0 | $-10$ |

Then, the solutions are $x_1 = 0,\ x_2 = 3$ and $Max.Z = Z_1 \times Z_2 = \frac{8}{7}$. Since all $\Delta_j \le 0$, then the solution is optimal.

**Example 4. 2:**

The following QFPP is taken into account:

$$Max.Z = \frac{(6x_1+4x_2+2x_3)(x_1+3x_2+2x_3+1)}{(3x_1+2x_2+x_3+2)(3x_1+2x_2+5)}$$

Subject to:

$$x_1 + 3x_2 + 2x_3 \le 9$$
$$3x_1 + 2x_2 + x_3 \le 8$$
$$2x_1 + x_2 + 3x_3 \le 7$$
$$x_1, x_2, x_3 \ge 0$$

**Solution:** Similar to example **4.1**, the solutions are $x_1 = 0,\ x_2 = 3, x_3 = 0$ and $Max.Z = \frac{15}{11}$. Since all $\Delta_j \le 0$, then the solution is optimal.

## 5. Conclusion

This study presented a quadratic fractional objective function specialization of the RBM-Technique. The algorithm selected a direction to be developed. By optimizing the objective function in that direction, an optimal step was selected, which was agreed upon because the RBM-Technique dealt with variables more than $n = 10$. So far, the researchers in [6], aiming at finding the optimal solution, changed the QFPP to Quadratic Programming Problem and solved it.

## References

[1] Abdulrahim, B. K. (2013). Solving quadratic fractional programming problem via feasible direction development and modified simplex method. *Journal of Zankoy Sulaimani (JZS)*, *15*(2), 45-52, https://doi.org/10.17656/jzs.10245

[2] Abdulrahim, B. K. (2017). Using feasible directions to solve quadratic fractional programming problem. *Journal of Garmin University*, *4*(1), 56-71, https://doi.org/10.24271/garmian.6.

[3] Abdulrahim, B. K. (2017). Using pseudoaffinity to translation QFPP to LFPP. *Journal of Garmin University*, *4*(13), 164-177, https://doi.org/10.24271/garmian.134
[4] Beck, A., Ben-Tal, A., & Teboulle, M. (2006). Finding a global optimal solution for a quadratically constrained fractional quadratic problem with applications to the regularized total least squares. *SIAM Journal on Matrix Analysis and Applications*, *28*(2), 425-445, https://doi.org/10.1137/040616851
[5] Zhang, A. (2008). *Quadratic fractional programming problems with quadratic constraints*. Doctoral dissertation, Graduate School of Informatics, Kyoto University, Japan.
[6] Sulaiman, N., & Nawkhass, M. (2013). Solving quadratic fractional programming problem. *International Journal of Applied Mathematical Research*, *2*(2), 303-309, https://doi.org/10.14419/ijamr.v2i2.838

# حل مشكلة البرمجة الجزئية التربيعية بطريقة جديدة

**ميديا باوةخان مراخان , بسية كاكةولا عبدالرحيم , رزكار فريق محمود , أرام خليل إبراهيم , شوان عدنان علي**

*قسم الرياضيات ، كلية التربية ، جامعة كرميان ، إقليم كوردستان ، العراق*

**الملخص**

في هذا البحث، تم توسيع الخوارزميات المعطاة في مشكلة البرمجة الجزئية التربيعية (QFPP). وتم تطوير منهجية لحل مشكلة البرمجة الكسرية التربيعية، والتي تكون وظيفتها الموضوعية مشكلة البرمجة الكسرية غير الخطية والقيود خطية. استخدمت تقنية RBM لحل المشكلة وبنتيجة جيدة. تمت أيضًا مناقشة تطبيق الكمبيوتر لخوارزمياتنا من خلال حل المثال العددي المركب باستخدام إصدار MATLAB 2013a.