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Solving Partial Differential Equations by using Efficient Hybrid Transform Iterative Method

Ruaa Shawqi Ismael, Ali Al -Fayadh, Saad M. Salman

Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University, Baghdad, Iraq

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Corresponding Author:

Name: Ruaa Shawqi Ismael

E-mail: ruaashawqi1990@gmail.com

Tel:

ABSTRACT

The aim of this article is to propose an efficient hybrid transform iteration method that combines the homotopy perturbation approach, the variational iteration method, and the Aboodh transform forsolving various partial differential equations. The Korteweg-de Vries (KdV), modified KdV, coupled KdV, and coupled pseudo-parabolic equations are given as examples to show how effective and practical the suggested method is. The obtained exact solitary solutions of the KdV equations as well as the exact solution of the coupled pseudo-parabolic equations are identified as a convergent series with easily calculable components. identified as a convergent series with easily calculable components.

When used to solve KdV , Wave like and Pseudo – Parabolic equations , the proposed method helps to avoid Problems that frequently arise when determining the Lagrange Multiplier and the difficult integration usedin the variation iteration method , as well as the need to use the transform convolution theorem.

حل المعادلات التفاضلية الجزئية باستخدام طريقة تكرار التحويل الهجينة الفعالة

رؤى شوقي أسماعيل ، علي حسن ناصر الفياض ، سعد محسن سلمان

قسم الرياضيات وتطبيقات الحاسوب ، كلية العلوم ، جامعة النهرين ، بغداد ، العراق

الملخص

الهدف من هذه المقالة هو اقتراح طريقة تكرار تحويل هجين فعالة تجمع بين طريقة الهوموتبي المضطرب والطريقة التغايرية التكرارية بالإضافة إلى تحويل Aboodh لحل المعادلات التفاضلية الجزئية المختلفة. إذ تم حل معادلات تفاضلية جزئية من النوع Korteweg-de Vries (KdV),modified KdV , coupled kdv ,and coupled pseudo parabolic equation لإظهار مدى فعالية الطريقة المقترحة وعملها. يتم تحديد الحلول الانفرادية الدقيقة التي تم الحصول عليها لمعادلات KdV بالإضافة إلى الحل الدقيق للمعادلات الزائدة المكافئة المقترنة كمتسلسلة متقاربة مع حدود قابلة للحساب بسهولة. عند استخدامها لحل معادلات KdV و Wave like و Pseudo - Paraboli تساعد الطريقة المقترحة على تجنب المشاكل التي تنشأ بشكل متكرر عند تحديد مضاعف Lagrange والتكامل الصعب المستخدم في طريقة التكرار المتغاير، بالإضافة إلى الحاجة إلى استخدام مبرهنة التفاف التحويل.

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1. Introduction

The application of diverse approaches for the numerical and analytical solution of partial differential equations (PDEqs) has made significant advances in recent years. Differential equations (DEs) such as a linear and nonlinear Korteweg-de Vries (KdV) like equations are employed in a variety of physical and fluid dynamical applications. These equations represent both nonlinearity and dispersion, have received a lot of interest and served as the model equation for the development of soliton theory. In the study of nonlinear dispersive waves, the KdV equation appears. It was initially developed by Korteweg-de Vries [1] while studying long water waves in a canal with a finite depth. The nonlinear term would disappear in long wave problems in relatively shallow water and for extremely small amplitudes [2]. Due to the significance of this equation in many industrial and scientific applications, a great deal of effort has been devoted to understanding these equations. Other applications of fluid dynamics have been explored by [3]. On the other hand, the parabolic equation appears in a number of applications of mathematics, including heat conduction, turbulence phenomena, and flow by the use of a shock wave moving through a viscous fluid, such as dynamics modelling. Many physical and engineering processes are modeled by the pseudo-parabolic equation [4-7]. The linear pseudo-parabolic equation is additionally referred to as a Sobolev type equation since S.L. Sobolev [8] was the first to study it in 1954. These equations serve as significant instances of PDEs since they include a third order mixed derivative with regard to both time and space. They are used to describe wave motion, which is crucial in many other branches of engineering and research in addition to hydrodynamics [9]. The equation for the one-dimensional pseudo-parabola was developed in [10].

Numerous scientific and engineering facts can be explained by nonlinear partial differential equations (NLPDEs). The availability of closed approximation solutions to nonlinear problems is frequently helpful in scientific applications. There are several different numerical and analytical techniques that can be used to solve the NLPDEs. Because numerical solutions have limitations, the analytical solution for a particular PDE is always preferred., which cannot give us much information on the qualitative behavior of systems. However, It is frequently impossible to achieve an accurate analytical solution to the equations for the systems that PDEs describe because they are too large or complex, or because many of the modelled issues have led to NLPDEs, for which it is difficult to find exact solutions. Numerical methods, iterative methods, perturbation methods, homotopy-based methods, etc. are well-known methodologies for approximating solutions of coupled systems of differential equations. Each method has advantages and drawbacks of its own. Discretization is employed in numerical approaches, which has an impact on accuracy as well as it required the most time and computing efforts. Many scholars have developed various numerical techniques over time These nonlinear equations require precise approximation solutions, which, despite the shortcomings of the numerical methods [11-17].

On the other hand, the iterative approaches generate a series of approximations for the solution. They produced sequential approximations using their initial guess. Computers can be effective for iteratively solving equations since these approaches require repeatedly performing the same operation. Finding an approximate solution to differential equations is a key part of iterative approaches. PDEs have been solved using a range of iterative techniques, including the Adomian decomposition method (ADM) [19], the variational iteration method (VIM), and the homotopy perturbation method (HPM) [18] These techniques produce a fast convergence to accuracy of either an approximation or perfect answer by employing terms from an infinite series. The evaluation of small parameters, the use of Adomian polynomials, and the computation of the Lagrange multiplier (LM) are some of the disadvantages of these methods. To improve the work of these iterative methods and avoid some of the limitations for solving various NLPDEQs and integral equations, a number of researchers are working very hard to combine various iterative techniques with transformations [21–36]. Since nonlinearity does occur, Possibly not always. simple using only the well-known integral transformations to quickly solve nonlinear equations. In addition, it should be noted, nonetheless, that the majority of the iterative procedures currently in use contain flaws, such as needless Transformation, linearization, variable discretization, or the use of constraining assumptions. Therefore, a significant number of approximate exact solutions were consequently produced by merging an appropriate integral transform with other iterative techniques.

Many researchers have employed different approaches to solve KdV equations. For new coupled modified KdV equations, Fan [37] worked on the extended tanh-function approach and symbolic computing to achieve, four different types of soliton solutions. The decomposition method was used by Raslan et al. [38] to find the soliton solutions for the coupled modified KdV equations. The homotopy analysis method (HAM) was applied to obtain the approximate solution of the modified KdV system [39]. ADM was implemented for obtaining the approximate solution of coupled modified KdV equations [40] .The effectiveness and accuracy of differential transform method (DTM) for proposed equations were proved by Kangalgil et al. [41]. Coupled modified KdV equation systems were

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numerically solved by Arife et al. using HAM [42]. In order to find approximate analytic solutions for coupled modified KdV equations, Zhong et al. employed the VIM [43]. Ao Zhu and Chengyang Fan [44] applied the ADM and symbolic computation system to obtain explicit exact solitary solutions of the modified KdV equation.

The objective of the current work is to extend the application of Aboodh integral transform merging with mixed iterative methods and analysis the performance of the proposed approach for addressing linear and nonlinear problems such as the KdV and the coupled modified KdV like equations; and to compare the obtained solutions to the exact solutions. The suggested method called efficient hybrid transform iterative method (EHTIM), is developed for solving the KdV, modified KdV, coupled KdV, and coupled pseudo-parabolic equations. This method incorporates the Aboodh transform [45], VIM, and HPM. The method has significance because it does not employ the integral portion or the convolution theorem, which are typically employed in VIM to determine the Lagrange multiplier, then reducing the time and calculationsThe exact solution, if it exists, is always obtained using this method, even though a small number of iterations can be employed for numerical purposes with a high degree of precision in the form of a quickly converging series. The method is easy to comprehend because it doesn't call for any presumptions that alter the problem's physical nature, as those that call for discretization, linearization, or minor components. The findings show that the suggested strategy is efficient, dependable, precise, and adaptable.

2. Some Preliminaries

In this part, an overview of all the components of the combined suggested EHTIM will be given.

2.1 Variational Iteration Method (VIM)

One of the most well-known and important methods for solving linear and nonlinear equations is the VIM.. The NLPDE will be used to illustrate the VIM concept [20]:

$L\mathbb{u} + \mathrm{N}\mathbb{u} = Q(x)$ ... (1)

where $Q(x)$ is an analytical function; L and N are a linear and a nonlinear operators respectively. The correction functional to the VIM for Eq. (1) is given by:

$\mathbb{u}_{\mathrm{r}+1}(x) = \mathbb{u}_{\mathrm{r}}(x) + \int_0^x \lambda(\zeta)\left[L\mathbb{u}_{\mathrm{r}}(\zeta) + \mathrm{N}_{\tilde{\mathbb{u}}'_{\mathrm{r}}}(\zeta) - Q(\zeta)\right]d\zeta$ ... (2)

Where $\tilde{\mathbb{u}}'_{\mathrm{r}}$ is a restricted variant, i.e. $\delta\tilde{\mathbb{u}}'_{\mathrm{r}} = 0$, the index r denotes the rth approximation, and $\lambda(\zeta)$ is the LM that can be accurately determined by the variational theory. The approximation $\mathbb{u}_{\mathrm{r}+1}$, $\mathrm{r} \geq 0$ of u will be achieved by using any selected function $\mathbb{u}_0$. The LM can be determined by using the integration by parts and the solution is given by,

$\mathbb{u} = \lim_{\mathrm{r}\to\infty} \mathbb{u}_{\mathrm{r}}$ ... (3)

2.2 Aboodh Transform ($\mathcal{AT}$)

For functions of an exponential order $\mathscr{f}(t)$ over the set $\mathfrak{P}$ given by [45]

$\mathfrak{P} = \{\mathscr{f};\ |\mathscr{f}(t)| < Ke^{|t|\beta_j}\ , t \in (-1)^j \times [0,\infty[\ , j = 1,2;\ \mathcal{K}, \beta_1, \beta_2 > 0\}$ ... (4)

where β_1, β_2 may be finite or infinite numbers and the constant $\mathcal{K}$ must be a finite number for a given function in $\mathfrak{P}$. the operator $\boldsymbol{\mathcal{A}}(:)$ is used to represent and defined $\mathcal{AT}$ as follows

$\boldsymbol{\mathcal{A}}[\mathscr{f}(t)] = \frac{1}{\mathbb{v}}\int_0^\infty \mathscr{f}(t)e^{-\mathbb{v}t}\,dt = \boldsymbol{\mathcal{A}}(\mathbb{v})\ ,\ t \geq 0\ ;\ -\beta_1 \leq \mathbb{v} \leq \beta_2$ . ... (5)

The basic properties of $\mathcal{AT}$ are given below, If $\boldsymbol{\mathcal{A}}(\mathbb{v})$ is the $\mathcal{AT}$ of $\mathscr{f}(t)$, then

1. $\boldsymbol{\mathcal{A}}[1] = \frac{1}{\mathbb{v}^2}$ .... (6)
2. $\boldsymbol{\mathcal{A}}[t] = \frac{1}{\mathbb{v}^3}$ .... (7)
3. $\boldsymbol{\mathcal{A}}[t^n] = \frac{n!}{\mathbb{v}^{n+2}}$ ... (8)
4. $\boldsymbol{\mathcal{A}}[e^{ct}] = \frac{1}{\mathbb{v}^2 - c\mathbb{v}}$ ... (9)
5. $\boldsymbol{\mathcal{A}}[a\mathscr{f}_1(t) + b\mathscr{f}_2(t)] = a\boldsymbol{\mathcal{A}}[\mathscr{f}_1(t)] + b\boldsymbol{\mathcal{A}}[\mathscr{f}_2(t)]$ ... (10)
6. $\boldsymbol{\mathcal{A}}[t\,\mathscr{f}(t)] = \left(-\frac{d}{d\mathbb{v}} - \frac{1}{\mathbb{v}}\right)\boldsymbol{\mathcal{A}}(\mathbb{v})$ ... (11)
7. $\boldsymbol{\mathcal{A}}[e^{ct}h(t)] = \frac{\mathbb{v}-c}{\mathbb{v}}\boldsymbol{\mathcal{A}}(\mathbb{v} - c)$ ... (12)
8. $\boldsymbol{\mathcal{A}}[sin(ct)] = \frac{c}{\mathbb{v}(\mathbb{v}^2+c^2)}$ .... (13)
9. $\boldsymbol{\mathcal{A}}[cos(ct)] = \frac{1}{(\mathbb{v}^2+c^2)}$ .... (14)
10. $\boldsymbol{\mathcal{A}}[sinh(ct)] = \frac{c}{\mathbb{v}(\mathbb{v}^2-c^2)}$ ... (15)
11. $\boldsymbol{\mathcal{A}}[cosh(ct)] = \frac{1}{(\mathbb{v}^2-c^2)}$ ... (16)
12. $\boldsymbol{\mathcal{A}}[\mathscr{f}'(t)] = \mathbb{v}\boldsymbol{\mathcal{A}}(\mathbb{v}) - \frac{\mathscr{f}(0)}{\mathbb{v}}$ ... (17)
13. $\boldsymbol{\mathcal{A}}[\mathscr{f}''(t)] = \mathbb{v}^2\boldsymbol{\mathcal{A}}(\mathbb{v}) - \frac{\mathscr{f}'(0)}{\mathbb{v}} - \mathscr{f}(0)$ .... (18)
14. $\boldsymbol{\mathcal{A}}[\mathscr{f}^{(n)}(t)] = \mathbb{v}^{(n)}\boldsymbol{\mathcal{A}}(\mathbb{v}) - \sum_{i=0}^{n-1}\frac{\mathscr{f}^{(i)}(0)}{\mathbb{v}^{2-n+i}}$ ... (19)
15. $\boldsymbol{\mathcal{A}}[t\mathscr{f}'(t)] = -\frac{d}{d\mathbb{v}}\left[\mathbb{v}\boldsymbol{\mathcal{A}}(\mathbb{v}) - \frac{\mathscr{f}(0)}{\mathbb{v}}\right] - \frac{1}{\mathbb{v}}\left(\mathbb{v}\boldsymbol{\mathcal{A}}(\mathbb{v}) - \frac{\mathscr{f}(0)}{\mathbb{v}}\right)$ ... (20)
16. $\boldsymbol{\mathcal{A}}[t^2\mathscr{f}'(t)] = \mathbb{v}\frac{d^2\boldsymbol{\mathcal{A}}(\mathbb{v})}{d\mathbb{v}^2} + 2\frac{d\boldsymbol{\mathcal{A}}(\mathbb{v})}{d\mathbb{v}} - 2\frac{\mathscr{f}(0)}{\mathbb{v}^3}$ ... (21)
17. $\boldsymbol{\mathcal{A}}[t\,\mathscr{f}''(t)] = -\frac{d}{d\mathbb{v}}\left[\mathbb{v}^2\boldsymbol{\mathcal{A}}(\mathbb{v}) - \frac{\mathscr{f}'(0)}{\mathbb{v}} - \mathscr{f}(0)\right] - \frac{1}{\mathbb{v}}\left(\mathbb{v}^2\boldsymbol{\mathcal{A}}(\mathbb{v}) - \frac{\mathscr{f}'(0)}{\mathbb{v}} - \mathscr{f}(0)\right)$ .. (22)
18. $\boldsymbol{\mathcal{A}}[t^2\,\mathscr{f}''(t)] = \mathbb{v}^2\frac{d^2\boldsymbol{\mathcal{A}}(\mathbb{v})}{d\mathbb{v}^2} + 4\mathbb{v}\frac{d\boldsymbol{\mathcal{A}}(\mathbb{v})}{d\mathbb{v}} + 2\boldsymbol{\mathcal{A}}(\mathbb{v}) - 2\frac{\mathscr{f}'(0)}{\mathbb{v}^3}$ (23)

For the partial derivative, the $\mathcal{AT}$ is given by [46],

19. $\boldsymbol{\mathcal{A}}[\mathbb{u}(x,t)] = \boldsymbol{\mathcal{A}}(x,\mathbb{v})$ .... (24)
20. $\boldsymbol{\mathcal{A}}\left[\frac{\partial\mathbb{u}(x,t)}{\partial t}\right] = \mathbb{v}\boldsymbol{\mathcal{A}}(x,\mathbb{v}) - \frac{\mathbb{u}(x,0)}{\mathbb{v}}$ ... (25)
21. $\boldsymbol{\mathcal{A}}\left[\frac{\partial^2\mathbb{u}(x,t)}{\partial t^2}\right] = \mathbb{v}^2\boldsymbol{\mathcal{A}}(x,\mathbb{v}) - \frac{\frac{\mathbb{u}(x,0)}{\partial t}}{\mathbb{v}} - \mathbb{u}(x,0)$ ... (26)
22. $\boldsymbol{\mathcal{A}}\left[\frac{\partial\mathbb{u}(x,t)}{\partial x}\right] = \boldsymbol{\mathcal{A}}'(x,\mathbb{v})$ ... (27)
23. $\boldsymbol{\mathcal{A}}\left[\frac{\partial^2\mathbb{u}(x,t)}{\partial x^2}\right] = \boldsymbol{\mathcal{A}}''(x,\mathbb{v})$ ... (28)
24. $\boldsymbol{\mathcal{A}}\left[\frac{\partial^k\mathbb{u}(x,t)}{\partial x^k}\right] = \boldsymbol{\mathcal{A}}^{(\mathrm{k})}(x,\mathbb{v})$ .. (29)

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2.3 Homotopy Perturbation Method (HPM)

A dependable and efficient method The HPM may be used to determine the precise or approximative solutions of DEs [18]. In order to provide an example, consider the following nonlinear system with the boundary conditions. the HPM and to demonstrate its
fundamental idea,

$L(\mathbb{u}) + \mathrm{N}(\mathbb{u}) - \mathcal{Q}(s) = 0\ ,\ s \in \boldsymbol{\Phi}$ ... (30)

$\mathfrak{P}\left(\mathbb{u}, \frac{\partial \mathbb{u}}{\partial n}\right) = 0\ ,\ s \in \boldsymbol{\Pi}$ ... (31)

where $\mathcal{Q}(s)$ is an analytical function, *A linear operator is L, and a nonlinear operator is N.*; $\frac{\partial \mathbb{u}}{\partial n}$ is the differentiation of $\mathbb{u}$ with respect to n, and $\boldsymbol{\Pi}$ is the boundary of the domain $\boldsymbol{\Phi}$

The following homotopy $\mathbb{v}(\mathbb{r}, \mathbb{p})$ may be constructed by applying the homotopy technique to Eq. (30),

$\mathbb{v}(\mathbb{r}, \mathbb{p}): \boldsymbol{\Phi} \times [0,1] \to \mathbb{R}$ ... (32)

which satisfies,

$\mathbb{W}(\mathbb{v}, \mathbb{p}) = (1 - \mathbb{p})[L(\mathbb{v}) - L(\mathbb{u}_0)] + \mathbb{p}[L(\mathbb{v}) + \mathrm{N}(\mathbb{v}) - \mathcal{Q}(s)] = 0,\ s \in \boldsymbol{\Phi}$ ... (33)

where $\mathbb{p} \in [0,1]$ which increases from 0 to 1, $\mathbb{R}$ is the set of real numbers, and $\mathbb{u}_0$ is an initial approximate solution of Eq (30), which satisfies the boundary conditions Eq. (31). Clearly, from Eq. (33) we have

$\mathbb{W}(\mathbb{v}, 0) = L(\mathbb{v}) - L(\mathbb{u}_0) = 0$ ... (34)

and

$\mathbb{W}(\mathbb{v}, 1) = L(\mathbb{v}) + \mathrm{N}(\mathbb{v}) - \mathcal{Q}(s) = 0.$ ... (35)

Assume that the following power series in $\mathbb{p}$ can be utilized to clarify the solution to Eq. (33), and $\mathbb{p} \in [0,1]$ as a small parameter:

$\mathbb{v} = \mathbb{v}_0 + \mathbb{p}\mathbb{v}_1 + \mathbb{p}^2\mathbb{v}_2 + \mathbb{p}^3\mathbb{v}_3 + \cdots$ ..(36)

With $\mathbb{p} = 1$, the solution $\mathbb{u}$ to Eq.(33) is given as;

$\mathbb{u} = \lim_{\mathbb{p}\to 1} \mathbb{v} = \mathbb{v}_0 + \mathbb{v}_1 + \mathbb{v}_2 + \mathbb{v}_3 + \cdots$ ... (37)

Most of the time, the series in Eq. (37) converges, but the pace of convergence varies depending on the nonlinear operator.$\mathrm{N}(\mathbb{v})$.

3. The Proposed Efficient Hybrid Transform Iterative Method for Solving Partial Differential Equations

(EHTIM)

The Aboodh transform (($\mathcal{AT}$)), the VIM, and the HPM are all combined to create the recommended EHTIM. The sides of a specific DE are first each given the ($\mathcal{AT}$). The Lagrange multiplier (LM) will be multiplied by the resulting equation to produce the recurrence relation. The (LM) is then produced by constraining the discovered recurrence relation. This method does not require the often employed convolution theorem or integral part. used to find (LM) in VIM.

applying $\mathcal{AT}$ of Eq.(1) result in,

$\mathcal{A}\left[L\mathbb{u} + \mathrm{N}\mathbb{u} - \mathcal{Q}(x)\right] = 0$ ... (38)

Multiplying Eq.(38) by (LM) $\lambda(\mathbb{v})$, we get

$\lambda(\mathbb{v})\mathcal{A}\left[L\mathbb{u} + \mathrm{N}\mathbb{u} - \mathcal{Q}(x)\right] = 0.$ ... (39)

The following recurrence relation is utilized to compute the value of (LM),

$\mathbb{u}_{m+1}(x, \mathbb{v}) = \mathbb{u}_m(x, \mathbb{v}) + \lambda(\mathbb{v})\mathcal{A}\left[L\mathbb{u} + \mathrm{N}\mathbb{u} - \mathcal{Q}(x)\right]$ ... (40)

By using the optimality criterion, the (LM), $\lambda(\mathbb{v})$ is determined, and

$\frac{\delta \mathbb{u}_{m+1}(x,\mathbb{v})}{\delta \mathbb{u}_m(x,\mathbb{v})} = 0.$ ... (41)

We get to $\lambda(\mathbb{v}) = -\frac{1}{\mathbb{v}^2}$by assuming $L = \frac{\partial}{\partial x^2}$to be a linear differential operator. The value of (LM) and the inverse of AT in Eq. (40) will be used to determine the approximate answer.

$\mathbb{u}_{m+1}(x, \mathbb{v}) = \mathbb{u}_m(x, \mathbb{v}) + \mathcal{A}^{-1}\left[-\frac{1}{\mathbb{v}^2}\left[L\mathbb{u} + \mathrm{N}\mathbb{u} - \mathcal{Q}(x)\right]\right],\ m = 0,1,2,3, \dots$ ... (42)

The HPM for nonlinear terms can be expressed as follows,

$\mathrm{N}(\mathbb{u}) = \sum_{j=0}^{\infty} \mathbb{p}^j \mathrm{H} = \mathrm{H}_0 + \mathbb{p}\mathrm{H}_1 + \mathbb{p}^2\mathrm{H}_2 + \cdots$ ... (43)

The formula below can be used to calculate the He's polynomial, $\mathrm{H}_m{}'$

$\mathrm{H}_m(\mathbb{u}_0 + \mathbb{u}_1 + \mathbb{u}_2 + \cdots + \mathbb{u}_m) = \frac{1}{m!}\frac{\partial^m}{\partial \mathbb{p}^m}\left[\mathrm{N}\left(\sum_{j=0}^{\infty} \mathbb{p}^j \mathbb{u}_j\right)\right]_{\mathbb{p}=0},\ m = 0,1,2,3, \dots$ ... (44)

The following approximations can be computed by comparing the coefficients of same powers of $\mathbb{p}$,

$\mathbb{p}^0 = \mathbb{u}_0(x, t)$ ... (45)

$\mathbb{p}^1 = \mathbb{u}_1(x, t) = -\mathcal{A}^{-1}\left[\frac{1}{\mathbb{v}^2}\mathcal{A}\left[\mathrm{N}(\mathbb{u}_0(x,t)) - \mathrm{H}(\mathbb{u}_0(x,t))\right]\right]$ ... (46)

$\mathbb{p}^2 = \mathbb{u}_2(x, t) = -\mathcal{A}^{-1}\left[\frac{1}{\mathbb{v}^2}\mathcal{A}\left[\mathrm{N}(\mathbb{u}_1(x,t)) - \mathrm{H}(\mathbb{u}_1(x,t))\right]\right]$ .. (47)

$\mathbb{p}^3 = \mathbb{u}_3(x, t) = -\mathcal{A}^{-1}\left[\frac{1}{\mathbb{v}^2}\mathcal{A}\left[\mathrm{N}(\mathbb{u}_2(x,t)) - \mathrm{H}(\mathbb{u}_2(x,t))\right]\right]$ ... (48)

and so on then the exact solution

$\mathbb{u}_m(x, t) = \mathbb{u}_0 + \mathbb{u}_1 + \mathbb{u}_2 + \mathbb{u}_3 + \cdots$ ... (49)

4. Illustration Examples

Problem 1 [47]**:** Consider the homogeneous linear KdV equation with the initial condition,

$\mathbb{u}_t + \mathbb{u}_x + \mathbb{u}_{xxx} = 0,\ \mathbb{u}(x, 0) = e^{-x}\ ,\ 0 \le x \le 1,\ \ t > 0$ ... (50)

The discussion from section 3 is continued here. The recurrence relation displayed bellow is obtained by applying the $\mathcal{AT}$ to both sides of Eq. (50) and it is then multiplied by $\lambda(\mathbb{v})$,

$\mathcal{A}[\mathbb{u}_t + \mathbb{u}_x + \mathbb{u}_{xxx}] = 0$ ... (51)

Multiplying Eq. (51) by $\lambda(\mathbb{v})$, yields

$\lambda(\mathbb{v})\left[\mathcal{A}[\mathbb{u}_t + \mathbb{u}_x + \mathbb{u}_{xxx}]\right] = 0$ .... (52)

The relation of recurrence is

$\mathbb{u}_{m+1}(x, \mathbb{v}) = \mathbb{u}_m(x, \mathbb{v}) + \lambda(\mathbb{v})\left[\mathcal{A}[\mathbb{u}_t + \mathbb{u}_x + \mathbb{u}_{xxx}]\right]$ .... (53)

As a result of applying $\mathcal{AT}$ depending on how the independent variable has changed $\mathbb{u}_m$ both sides of the Eq. (53), we obtain

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$\delta \mathbb{u}_{m+1}(x,\mathbb{v}) =$
$\delta \mathbb{u}_{m}(x,\mathbb{v}) + \lambda(\mathbb{v})\left[\delta\left(\mathbb{u}_{m}(x,\mathbb{v}) - \mathbb{u}(x,0)\right) - \boldsymbol{\mathcal{A}}(\mathbb{u}_x - \mathbb{u}_{xxx})\right] \ldots (54)$
$\delta \mathbb{u}_{m+1}(x,\mathbb{v}) = \delta \mathbb{u}_{m}(x,\mathbb{v})\left(1 + \mathbb{v}\,\lambda(\mathbb{v})\right)$
when $\frac{\delta \mathbb{u}_{m+1}}{\delta \mathbb{u}_m} = 0,$
$1 + \mathbb{v}\,\lambda(\mathbb{v}) = 0$, then $\lambda(\mathbb{v}) = -\frac{1}{\mathbb{v}}.$
$\mathbb{u}_{m+1}(x,\mathbb{v}) = \mathbb{u}_{m}(x,\mathbb{v}) - \frac{1}{\mathbb{v}}\left[\boldsymbol{\mathcal{A}}[\mathbb{u}_t + \mathbb{u}_x + \mathbb{u}_{xxx}]\right] \ldots (55)$
Using the inverse of $\boldsymbol{\mathcal{AT}}$ yields,
$\mathbb{u}_{m+1}(x,\mathbb{v}) = \mathbb{u}_{m}(x,\mathbb{v}) - \boldsymbol{\mathcal{A}}^{-1}\left[\frac{1}{\mathbb{v}}\left[\boldsymbol{\mathcal{A}}[\mathbb{u}_t + \mathbb{u}_x + \mathbb{u}_{xxx}]\right]\right] \ldots (56)$
Utilizing the HPM, we obtain
$\mathbb{u}_0 + \mathbb{p}\mathbb{u}_1 + \mathbb{p}^2\mathbb{u}_2 + \mathbb{p}^3\mathbb{u}_3 + \cdots = \mathbb{u}_m(x,t) + \mathbb{p}\boldsymbol{\mathcal{A}}^{-1}\left[\frac{1}{\mathbb{v}}[\boldsymbol{\mathcal{A}}\{(-\mathbb{u}_{0x} - \mathbb{u}_{0xxx}) + \mathbb{p}(-\mathbb{u}_{1x} - \mathbb{u}_{1xxx}) + \mathbb{p}^2(-\mathbb{u}_{2x} - \mathbb{u}_{2xxx}) + \mathbb{p}^3(\mathbb{u}_{3x} - \mathbb{u}_{3xxx}) + \cdots\}]\right] \ldots (57)$
After equating $\mathbb{p}$ with the same powers on both sides, the He's polynomials are,
$\mathbb{p}^0: \mathbb{u}_0 = \mathbb{u}_0(x,t) = e^{-x},$
$\mathbb{p}^1: \mathbb{u}_1 = \boldsymbol{\mathcal{A}}^{-1}\left[\frac{1}{\mathbb{v}}[\boldsymbol{\mathcal{A}}[-\mathbb{u}_{0x}^2\mathbb{u}_{0x} - \mathbb{u}_{0xxx}]]\right] = 2te^{-x},$
$\mathbb{p}^2: \mathbb{u}_2 = \boldsymbol{\mathcal{A}}^{-1}\left[\frac{1}{\mathbb{v}}[\boldsymbol{\mathcal{A}}[-\mathbb{u}_{1x}^2\mathbb{u}_{1x} - \mathbb{u}_{1xxx}]]\right] = 4\frac{t^2}{2!}e^{-x},$
$\mathbb{p}^3: \mathbb{u}_3 = \boldsymbol{\mathcal{A}}^{-1}\left[\frac{1}{\mathbb{v}}[\boldsymbol{\mathcal{A}}[-\mathbb{u}_{2x}^2\mathbb{u}_{2x} - \mathbb{u}_{2xxx}]]\right] = 8\frac{t^3}{3!}e^{-x},$
and so on. Then
$\mathbb{u}_m(x,t) = e^{-x} + 2te^{-x} + 4\frac{t^2}{2!}e^{-x} + 8\frac{t^3}{3!}e^{-x} + \cdots \ldots (58)$
so that the exact solution by using the Taylor series is given by,
$\mathbb{u}(x,t) = e^{-x+2t} \ldots (59)$
Problem 2 [44]**:** Consider the modified KdV equation with the initial condition
$\mathbb{u}_t + \mathbb{u}^2\mathbb{u}_x + \mathbb{u}_{xxx} = 0,$
$\mathbb{u}(x,0) = \sqrt{6}\mathcal{k}\ sech(\mathcal{k}x) \ldots (60)$
where $\mathcal{k}$ is an arbitrary constant.
The discussion from section 3 is continued here. The recurrence relation displayed bellow is obtained by applying the $\boldsymbol{\mathcal{AT}}$ to both sides of Eq. (60) and it is then multiplied by $\lambda(\mathbb{v})$,
$\lambda(\mathbb{v})\left[\boldsymbol{\mathcal{A}}[\mathbb{u}_t + \mathbb{u}^2\mathbb{u}_x + \mathbb{u}_{xxx}]\right] = 0 \ldots (61)$
The recurrence relation is
$\mathbb{u}_{m+1}(x,\mathbb{v}) = \mathbb{u}_{m}(x,\mathbb{v}) + \lambda(\mathbb{v})\left[\boldsymbol{\mathcal{A}}[\mathbb{u}_t + \mathbb{u}^2\mathbb{u}_x + \mathbb{u}_{xxx}]\right] \ldots (62)$
As a result of applying $\boldsymbol{\mathcal{AT}}$ depending on how the independent variable has changed $\mathbb{u}_m$ both sides of the Eq. (62), we obtain
$\lambda(\mathbb{v}) = -\frac{1}{\mathbb{v}}.$
$\mathbb{u}_{m+1}(x,\mathbb{v}) = \mathbb{u}_{m}(x,\mathbb{v}) - \frac{1}{\mathbb{v}}\left[\boldsymbol{\mathcal{A}}[\mathbb{u}_t + \mathbb{u}^2\mathbb{u}_x + \mathbb{u}_{xxx}]\right] \ldots (63)$
Using the inverse of $\boldsymbol{\mathcal{AT}}$ yields,
$\mathbb{u}_{m+1}(x,\mathbb{v}) = \mathbb{u}_{m}(x,\mathbb{v}) - \boldsymbol{\mathcal{A}}^{-1}\left[\frac{1}{\mathbb{v}}\left[\boldsymbol{\mathcal{A}}[\mathbb{u}_t + \mathbb{u}^2\mathbb{u}_x + \mathbb{u}_{xxx}]\right]\right] \ldots (64)$
Utilizing the HPM and equating $\mathbb{p}$ with the same powers on both sides, the He's polynomials are;
$\mathbb{p}^0: \mathbb{u}_0 = \mathbb{u}_0(x,t) = \sqrt{6}\,\mathcal{k}\,\mathrm{sech}(\mathcal{k}x),$
$\mathbb{p}^1: \mathbb{u}_1 = \boldsymbol{\mathcal{A}}^{-1}\left[\frac{1}{\mathbb{v}}[\boldsymbol{\mathcal{A}}[-\mathbb{u}_{0x}^2\mathbb{u}_{0x} - \mathbb{u}_{0xxx}]]\right] = \sqrt{6}\,\mathcal{k}^4\,\mathrm{sech}(\mathcal{k}x)\tanh(\mathcal{k}x)\,t = \sqrt{6}\,\mathcal{k}^4\frac{\sinh(\mathcal{k}x)}{(\cosh(\mathcal{k}x))^2}t,$
$\mathbb{p}^2: \mathbb{u}_2 = \boldsymbol{\mathcal{A}}^{-1}\left[\frac{1}{\mathbb{v}}[\boldsymbol{\mathcal{A}}[-\mathbb{u}_{1x}^2\mathbb{u}_{1x} - \mathbb{u}_{1xxx}]]\right] = \sqrt{6}\,\mathcal{k}^7\frac{(-2+(\cosh(\mathcal{k}x))^2)}{(\cosh(\mathcal{k}x))^3}\frac{t^2}{2!},$
$\mathbb{p}^3: \mathbb{u}_3 = \boldsymbol{\mathcal{A}}^{-1}\left[\frac{1}{\mathbb{v}}[\boldsymbol{\mathcal{A}}[-\mathbb{u}_{2x}^2\mathbb{u}_{2x} - \mathbb{u}_{2xxx}]]\right] = \sqrt{6}\,\mathcal{k}^{10}\frac{\sinh(\mathcal{k}x)(-6+(\cosh(\mathcal{k}x))^2)}{(\cosh(\mathcal{k}x))^4}\frac{t^3}{3!},$
and so on. Then
$\mathbb{u}_m(x,t) = \sqrt{6}\,\mathcal{k}\,\mathrm{sech}(\mathcal{k}x) + \sqrt{6}\,\mathcal{k}^4\frac{\sinh(\mathcal{k}x)}{(\cosh(\mathcal{k}x))^2}t + \sqrt{6}\,\mathcal{k}^7\frac{(-2+(\cosh(\mathcal{k}x))^2)}{(\cosh(\mathcal{k}x))^3}\frac{t^2}{2!} + \sqrt{6}\,\mathcal{k}^{10}\frac{\sinh(\mathcal{k}x)(-6+(\cosh(\mathcal{k}x))^2)}{(\cosh(\mathcal{k}x))^4}\frac{t^3}{3!} + \cdots \quad (65)$
so that the exact solution by using the Taylor series is given by,
$\mathbb{u}(x,t) = \sqrt{6}\,\mathcal{k}\,\mathrm{sech}\left(\mathcal{k}(x - \mathcal{k}^2t)\right) \ldots (66)$
Furthermore, by modifying the argument in Eq. (66) by including a constant, we can obtain more exact solutions. This means that we can present the exact solutions as,
$\mathbb{u}(x,t) = \sqrt{6}\,\mathcal{k}\,\mathrm{sech}(\mathcal{k}(x - \mathcal{k}^2t) + \mathfrak{h}),$
where $\mathfrak{h}$ is a constant ... (67)
Problem 3 [40] **:** Consider the generalized coupled KdV equations with the initial conditions
$\mathbb{u}_t + 3\mathbb{u}^2\mathbb{u}_x - 3(\mathbb{u}\mathbb{w})_x = \frac{1}{2}\mathbb{u}_{xxx} + \frac{3}{2}\mathbb{w}_{xx} - 3\hbar\mathbb{u}_x, \ldots (68)$
$\mathbb{w}_t - 3\mathbb{w}\mathbb{w}_x + 3\mathbb{u}_x\mathbb{w}_x - 3\mathbb{u}^2\mathbb{w}_x = -\mathbb{w}_{xxx} + 3\hbar\mathbb{w}_x, \ldots .(69)$
$\mathbb{u}(x,0) = \frac{\mathcal{b}_1}{2\mathcal{k}} + \mathcal{k}\tanh(\mathcal{k}x),\ \mathbb{w}(x,0) = \frac{\mathcal{k}}{2}\left(1 + \frac{\mathcal{k}}{\mathcal{b}_1}\right) + \mathcal{b}_1\tanh(\mathcal{k}x). \ldots (70)$
Similar to the above problems, we follow the discussion from section 3 where $\mathcal{b}_1 = \mathcal{k} = \hbar = 1.$
The recurrence relation displayed bellow is obtained by applying the $\boldsymbol{\mathcal{AT}}$ to both sides of Eqs. (68-69) and it is then multiplied each of these equations by the Lagrange multipliers $\lambda_1(\mathbb{v})$ and $\lambda_2(\mathbb{v})$ respectively.

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$$\lambda_1(v)\left[\mathcal{A}\left[u_t+3u^2u_x-3(uw)_x-\frac{1}{2}u_{xxx}-\frac{3}{2}w_{xx}+3\hbar u_x\right]\right]=0\ \ldots..(71)$$

$$\lambda_2(v)\left[\mathcal{A}[w_t-3ww_x+3u_xw_x-3u^2w_x+w_{xxx}-3\hbar w_x]\right]=0\ \ldots(72)$$

The recurrence relation is

$$u_{m+1}(x,v)=u_m(x,v)+\lambda_1(v)\left[\mathcal{A}\left[u_t+3u^2u_x-3(uw)_x-\frac{1}{2}u_{xxx}-\frac{3}{2}w_{xx}+3\hbar u_x\right]\right]\ldots(78)$$

$$w_{m+1}(x,v)=w_m(x,v)+\lambda_2(v)\left[\mathcal{A}[w_t-3ww_x+3u_xw_x-3u^2w_x+w_{xxx}-3\hbar w_x]\right]\ldots(79)$$

Then $\lambda_1(v)=\lambda_2(v)\ -\frac{1}{v}$.

Using the inverse of $\mathcal{AT}$ yields,

$$u_{m+1}(x,v)=u_m(x,v)-\mathcal{A}^{-1}\left[\frac{1}{v}\left[\mathcal{A}\left[u_t+3u^2u_x-3(uw)_x-\frac{1}{2}u_{xxx}-\frac{3}{2}w_{xx}+3u_x\right]\right]\right]\ldots(80)$$

$$w_{m+1}(x,v)=w_m(x,v)+\mathcal{A}^{-1}\frac{1}{v}\left[\frac{1}{v}\left[\mathcal{A}[w_t-3ww_x+3u_xw_x-3u^2w_x+w_{xxx}-3w_x]\right]\right]\ldots(81)$$

Utilizing the HPM and using the same powers on both sides of Eq to equate p with. (80), the He's polynomials are;

$$p^0: u_0=u_0(x,t)=\frac{1}{2}+\tanh(x),$$

$$p^1: u_1=\mathcal{A}^{-1}\left[\frac{1}{v}\left[\mathcal{A}\left[-3u_0^2u_{0x}+3(u_0w_0)_x+\frac{1}{2}u_{0_{xxx}}+\frac{3}{2}w_{0_{xx}}+3u_{0_x}\right]\right]\right]=\frac{t}{4}[-1+\tanh^2(x)],$$

$$p^2: u_2=\mathcal{A}^{-1}\left[\frac{1}{v}\left[\mathcal{A}\left[-3u_1^2u_{1x}+3(u_1w_1)_x+\frac{1}{2}u_{1_{xxx}}+\frac{3}{2}w_{1_{xx}}+3u_{1_x}\right]\right]\right]=\frac{t^2}{16}[-\tanh(x)+\tanh^3(x)],$$

and so on. Then

$$u_m(x,t)=\frac{1}{2}+\tanh(x)+\frac{t}{4}[-1+\tanh^2(x)]+\frac{t^2}{16}[-\tanh(x)+\tanh^3(x)]+\cdots\ldots(82)$$

so that the exact solution by using the Taylor series is given by,

$$u(x,t)=\frac{1}{2}+\tanh(x+kt)\ \ldots(83)$$

Similarly that,

$$w(x,t)=\frac{1}{2}+\tanh(x+kt)\ \ldots.(84)$$

Problem 4 [48]**:** Consider the following homogenous of coupled-pseudo-parabolic Equation,

$$u_t+\frac{1}{x}\frac{\partial}{\partial x}(xu_x)-\frac{1}{x}\frac{\partial^2}{\partial x\partial t}(xu_x)+w=0\ ,\ \ldots.(85)$$

$$w_t+\frac{1}{x}\frac{\partial}{\partial x}(xw_x)-\frac{1}{x}\frac{\partial^2}{\partial x\partial t}(xw_x)+u=0\ ,\ldots.(86)$$

$$u(x,0)=x^2,\ \ w(x,0)=x^2.\ \ldots\ (87)$$

we will follow the same procedure as above. The recurrence relation displayed bellow is obtained by applying the $\mathcal{AT}$ to both sides of Eqs. (85-86) and it is then multiplied each of these equations by the Lagrange multipliers $\lambda_1(v)$ and $\lambda_2(v)$ respectively.

$$\lambda_1(v)\left[\mathcal{A}\left[u_t+\frac{1}{x}\frac{\partial}{\partial x}(xu_x)-\frac{1}{x}\frac{\partial^2}{\partial x\partial t}(xu_x)+w\right]\right]=0\ \ldots(88)$$

$$\lambda_2(v)\left[\mathcal{A}\left[w_t+\frac{1}{x}\frac{\partial}{\partial x}(xw_x)-\frac{1}{x}\frac{\partial^2}{\partial x\partial t}(xw_x)+u\right]\right]=0\ \ldots.(89)$$

The recurrence relation is

$$u_{m+1}(x,v)=u_m(x,v)+\lambda_1(v)\left[\mathcal{A}\left[u_t+\frac{1}{x}\frac{\partial}{\partial x}(xu_x)-\frac{1}{x}\frac{\partial^2}{\partial x\partial t}(xu_x)+w\right]\right]\ldots.(90)$$

$$w_{m+1}(x,v)=w_m(x,v)+\lambda_2(v)\left[\mathcal{A}\left[w_t+\frac{1}{x}\frac{\partial}{\partial x}(xw_x)-\frac{1}{x}\frac{\partial^2}{\partial x\partial t}(xw_x)+u\right]\right]\ldots.(91)$$

As a result of applying $\mathcal{AT}$ on the variation with regard to the independent variable u_m on both sides of Eqs. (90-91), we obtain

$\lambda_1(v)=\lambda_2(v)=\ -\frac{1}{v}$.

Using the inverse of $\mathcal{AT}$ yields,

$$u_{m+1}(x,v)=u_m(x,v)-\mathcal{A}^{-1}\left[\frac{1}{v}\left[\mathcal{A}\left[u_t+\frac{1}{x}\frac{\partial}{\partial x}(xu_x)-\frac{1}{x}\frac{\partial^2}{\partial x\partial t}(xu_x)+w\right]\right]\right]\ldots(92)$$

$$w_{m+1}(x,v)=w_m(x,v)+\mathcal{A}^{-1}\left[\frac{1}{v}\left[\mathcal{A}\left[w_t+\frac{1}{x}\frac{\partial}{\partial x}(xw_x)-\frac{1}{x}\frac{\partial^2}{\partial x\partial t}(xw_x)+u\right]\right]\right]\ldots.(93)$$

Utilizing the HPM and equating p with the same powers on both sides of Eq. (92), the He's polynomials are;

$$p^0: u_0=u_0(x,t)=x^2,$$

$$p^1: u_1=\mathcal{A}^{-1}\left[\frac{1}{v}\left[\mathcal{A}\left[u_t+\frac{1}{x}\frac{\partial}{\partial x}(xu_{0x})-\frac{1}{x}\frac{\partial^2}{\partial x\partial t}(xu_{0x})+w_0\right]\right]\right]=4t-x^2t,$$

$$p^2: u_2=\mathcal{A}^{-1}\left[\frac{1}{v}\left[\mathcal{A}\left[u_t+\frac{1}{x}\frac{\partial}{\partial x}(xu_{1x})-\frac{1}{x}\frac{\partial^2}{\partial x\partial t}(xu_{1x})+w_1\right]\right]\right]=-4t-4t^2+\frac{x^2t^2}{2!},$$

$$p^3: u_3=\mathcal{A}^{-1}\left[\frac{1}{v}\left[\mathcal{A}\left[u_t+\frac{1}{x}\frac{\partial}{\partial x}(xu_{2x})-\frac{1}{x}\frac{\partial^2}{\partial x\partial t}(xu_{2x})+w_2\right]\right]\right]=2t^3-4t^2-4t+\frac{x^2t^3}{3!},$$

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and so on. Then the exact solution

$$\mathbb{u}_{\mathrm{m}}(x,t)=x^2-x^2t+\frac{x^2t^2}{2!}-\frac{x^2t^3}{3!}+\cdots \;\ldots(94)$$

so that the exact solution by using the Taylor series is given by,

$$\mathbb{u}(x,t)=x^2e^{-t} \;\ldots.(95)$$

Similarly that,

$$\mathbb{w}(x,t)=x^2e^{-t} \;\ldots..(96)$$

Conclusion

In this research, we determine exact solutions for the KdV equations and a coupled pseudo-parabolic equation. The KdV equations' solitary travelling wave solutions and the solution of the coupled pseudo-parabolic equation are generated using a hybrid method termed, efficient hybrid transform iterative method (EHTIM). The approach incorporates the HPM, VIM, and Aboodh transformation. The exact solutions were discovered as a rapidly convergent series of straightforward mathematical expressions. Additionally, The Aboodh's transform is utilised by the EHTIM, which is helpful for the computation of the Lagrange multiplier (LM), in contrast to the regular VIM and modified VIM processes. The method to calculate the LM does not require the convolution theorem or any integration in a recurrence relation. The suggested method has a further advantage over the decomposition method in that nonlinear problems can be resolved without the use of Adomian's polynomials. In addition, the proposed approach makes dealing with nonlinear terms more adaptive, flexible, and straightforward by using the He's polynomials that are computed using the HPM. Also, unlike some other techniques, the method does not require any assumptions that affect the issue's physical nature, such as those that involve linearization, discretization, or small elements. The collected results show how well the suggested strategy works, is dependable, accurate, and flexible in finding the right answers to the test questions.

References

[1] Korteweg, D. J. and de Vries, G. On the change of form of long waves advancing a rectangular canal and on a new type of long stationary waves, Phil. Mag., 39, pp. 422 – 443, (1895).

[2] Whitham, G. B. Linear and nonlinear waves. John Wiley & Sons, California, (1974).

[3] Shen, M. C. Asymptotic theory of unsteady three-dimensional waves in a channel of arbitrary cross section, SZAM. J. Appl. Math., 17, pp. 260-271, (1969).

[4] G.I. Barenblatt, I.P. Zheltov, I.N. Kochina, Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks, J. Appl. Math. Mech. 24 (1960) 1286–1303.

[5] P.J. Chen, M.E. Gurtin, On a theory of heat conduction involving two temperatures, Z. Angew. Math. Phys. 19 (1968) 614–627.

[6] T.W. Ting, Certain non-steady flows of second-order fluids, Arch. Rational Mech. Anal. 14 (1963) 1–26.

[7] D. Taylor, Research on Consolidation of Clays, Institute of Technology Press, Massachusetts, Cambridge, MA, 1952.

[8] S.L. Sobolev, A new problem in mathematical physics, Izv. Akad. Nauk SSSR Ser. Mat. 18 (1954) 3–50.

[9] F. Gao, J. Cui, G. Zhao, Weak Galerkin finite element methods for Sobolev equation, J. Comput. Appl. Math. 317 (2017) 188–202.

[10] Mesloub S. A nonlinear nonlocal mixed problem for a second order pseudoparabolic equation. J Math Anal Appl, 2006, 316: 189–209

[11] Hu H. Y. and Li Z. C. Collocation methods for Poisson's equation. Computer Methods in Applied Mechanics and Engineering, Vol. 195, No. 33, pp. 4139–4160, (2006).

[12] Thom´ee V. From finite differences to finite elements. A short history of numerical analysis of partial differential equations. Journal of Computational and Applied Mathematics, Vol. 128, No.1-2, pp. 1–54, (2001).

[13] Tatari M. and Dehghan M. On the solution of the nonlocal parabolic partial differential equations via radial basis functions. Applied Mathematical Modelling, Vol. 33, No. 3, pp. 1729–1738, (2009).

[14] Dehghan M. and Shokri A. A numerical method for two dimensional Schrodinger equation using collocation and radial basis functions. Computers & Mathematics with Applications, Vol. 54, No. 1, pp. 136–146, (2007).

[15] Ghada H. Ibraheem, Majeed A. Al-Jawary. The operational matrix of Legendre polynomials for solving nonlinear thin film flow problems. Alexandria Engineering Journal 59, pp. 4027–4033, (2020).

[16] Majeed A. AL-Jawary, Mustafa M. Azeez, Ghassan H. Radhi. Analytical and numerical solutions for the nonlinear Burgers and advection–diffusion equations by using a semi-analytical iterative method. Computers and Mathematics with Applications 76, pp. 155–171, (2018).

[17] Kumar S, Kumar A, Baleneu D. Two analytical method for time–fractional nonlinear coupled Boussinesq-Burger equations arises in propagation of shallow water waves. Nonlinear Dyn, 85:117, (2016).

[18] He J. Huan. Homotopy perturbation technique. Comput. Math. Appl. Mech. Eng., 178(3–4), pp.257-262, (1999).

[19] Adomian G. A. Solving frontier problems of Physics: The decomposition methods. Kluver Academic Publishers, Boston, 19 (1995).

[20] He J. Huan. Variational iteration method - a kind of nonlinear analytical technique: Some examples. Int. J. Nonlin. Mech., 34, pp.699-708, (1999).
[21] Marwa H. Al-Tai and Ali Al-Fayadh. Solving two dimensional coupled Burger's equations using transform variational iteration method. AIP Conf. Proc. 2414, 040030 (2023).
[22] Huda J. Kadhum and Ali Al-Fayadh. An efficient variational homotopy transform method for solving Schrödinger equation. *J. Phys.: Conf. Ser.* **2322**012044, (2022).
[23] Nadeem M. and Fengquan Li. Modified Laplace variational iteration method for analytical approach of Klein–Gordon and Sine–Gordon equations. Iranian Journal of Science and Technology, Transactions A: Science 43.4 :1933-1940, (2019).
[24] Emad K. J., Omar A., Mohammad A., and Ala'a M. Al-Faqih. An approximate analytical solution of the nonlinear Schrödinger equation with harmonic oscillator using homotopy perturbation method and Laplace-Adomian decomposition method. Advances in Mathematical Physics, vol. 2018, Article ID 6765021, 11 pages, 5 Dec. (2018).
[25] Marwa H. Al-Tai and Ali Al-Fayadh. Solving two dimensional coupled Burger's equations and Sine-Gordon equation using El-Zaki transform-variational iteration method. ANJS, vol.24 (2), June, pp. 41-47, (2021).
[26] Madani M. and Fathizadeh M. Homotopy perturbation algorithm using Laplace transformation. Nonlinear Science Letters A 1, pp. 263–267, (2010).
[27] Khan Y. and Wu Q., "Homotopy perturbation transform method for nonlinear equations using He's polynomials", Computers & Mathematics with Applications 59, pp. 622–628, (2010).
[28] Mohanned F. Kazem and Ali Al-Fayadh. Solving Fredholm integro-differential equation of fractional order by using Sawi homotopy perturbation method. J. Phys.: Conf. Ser. **2322**012056, (2022).
[29] Arun K. and Pankaj R. D. Solitary wave solutions of Schrödinger equation by Laplace–Adomian decomposition method. Physical Review and Research International, 3(4), pp.702–712, (2013).
[30] Kashuri A., Fundo A. and Kreku M. Mixture of a new integral transform and homotopy perturbation method for solving nonlinear partial differential equations. Advances in Pure Mathematics, 3, pp.317-323, (2013).
[31] Ali Al-Fayadh and Dina Saad. Laplace substitution–variational iteration method for solving Goursat problems involving mixed partial derivatives. American Journal of Mathematical and Computer Modelling, 4(1), 16, (2019).
[32] Ali Al-Fayadh and Hasan Ali Khawwan. Variational iteration transform method for solving Burger and coupled Burger's equations. ARPN Journal of Engineering and Applied Sciences, vol.12, no.23, Dec.(2017).
[33] Mohanned F. Kazem and Ali Al-Fayadh. Mixed homotopy integral transform method for solving non-linear integro-differential equation. ANJS, vol.25 (1), March, pp. 35-40, (2022).
[34] Asma A. Elbeleze , Adem Kılıcman, Bachok M. Taib. Modified homotopy perturbation method for solving linear second-order Fredholm integro–differential equations. Filomat 30:7, pp.1823–1831, (2016).
[35] Huda J. Kadhum and Ali Al-Fayadh. Solving Schrödinger equations using Kashuri and Fundo transform decomposition method. ANJS, vol.25 (2), June, pp. 25-28, , (2022).
[36] Nuruddeen R.I. and Aboodh K.S. Analytical solution for time-fractional diffusion equation by Aboodh decomposition method. International Journal of Mathematics and its Application, (2017).
[37] Fan E. G. Soliton solutions for a generalized Hirota-Satsuma coupled KdV equation and a coupled MKdV equation. Physics Letters A, Vol. 282, No. 1-2, pp. 18–22, (2001).
[38] Raslan K. R. The decomposition method for a Hirota-Satsuma coupled KdV equation and a coupled MKdV equation. International Journal of Computer Mathematics, Vol. 81, No. 12, pp. 1497–1505, (2004).
[39] Ghoreishi M., Ismail A. I., and Rashid A. The solution of coupled modified KdV system by the homotopy analysis method. TWMS Journal of Pure and Applied Mathematics, Vol. 3, No. 1, pp. 122–134, (2012).
[40] Soliman A.A. , M.A. Abdou. The decomposition method for solving the coupled modified KdV equations. Mathematical and Computer Modelling 47, pp. 1035–1041, (2008).
[41] Kangalgil F. and Ayaz F. Solitary wave solutions for Hirota-Satsuma coupled KDV equation and coupled mKDV equation by differential transform method. Arabian Journal for Science and Engineering. AJSE. Mathematics, Vol. 35, No. 2D, pp. 203–213, (2010).
[42] Arife A. S., Karimi Vanani S., and Yildirim A. Numerical solution of hirota-satsuma couple KdV and a coupled MKdV equation by means of homotopy analysis method. World Applied Sciences Journal, Vol. 13, No. 11, pp. 2271–2276, (2011).
[43] Guo-Zhong Z., Xi-Jun, X. Y., Jiang Yun,Z., and Di W. Approximate analytic solutions for a generalized Hirota-Satsuma coupled KdV equation and a coupled MKDV equation. Chinese Physics B, Vol. 19, No. 8, Article ID 080204, (2010).
[44] Ao Zhu and Chengyang Fan. Construction of exact solitary solutions of the modified KdV equation by Adomain decomposition method. ICIC Express Letters Part B: Applications, Volume 8, Number 8, August (2017).
[45] Aboodh K. S. The new integral transform "Aboodh Transform". Global Journal of Pure and

https://doi.org/10.25130/tjps.v29i3.1566

Applied Mathematics, ISSN 0973-1768, Vol. 9, No. 1, pp. 35-43, (2013).

[46] Aboodh K. S., Almardy I. A., Farah R. A., Ahmed M. Y., and Nuruddeen R. I. On the application of Aboodh transform to system of partial differential equations. BEST: Int. J. Humanities, Vol. 10, Issue. 2, pp. 33-46, (2022).

[47] Kaya D. On the solution of a Korteweg-de Varies equation by the decomposition method. Intern. J. Computer Math., Vol. 72, pp. 531 -539, (1999).

[48] Nadeem M. and Shao-Wen Y. Solving system of partial differential equations using variational iteration method with He's polynomials. J. Math. Computer Sci., 19, pp. 203–211. (2019).