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Existence and uniqueness of Solution for Boundary Value Problem of Fractional Order

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ABSTRACT

In this study, we investigate a class of fractional ordering and fractional derivative-based boundary value problems. $\alpha \in (3,4]$ and $\zeta \in (0, a]$. There are four boundary value requirements in this equation. The Banach fixed point theorem (Contraction mapping theorem) and the Schauder fixed point theorem are both used to arrive at the existence and uniqueness solution. Examples based on the fractional integral method and integral operator are used to illustrate our main points.

وجود ووحدانية الحل لمسألة القيمة الحدودية من الرتبة الكسرية

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الملخص

في هذه الدراسة، نقوم بالتحقيق في فئة من مسائل الترتيب الكسري ومسائل القيمة الحدودية القائمة على المشتقات الكسرية $\alpha\in(3,4]$ و $\zeta\in(0,a)$. هناك أربعة متطلبات للقيمة الحدية في هذه المعادلة. يتم استخدام نظرية باناخ للنقطة الثابتة (مبرهنة التطبيق الانكماشي) ومبرهنة النقطة الثابتة شاودر للوصول إلى وجود ووحدانية الحل. وتم استخدام الأمثلة المبنية على طريقة التكامل الكسري والمؤثر التكاملي لتوضيح النقاط الرئيسية.

1. Introduction

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Since it is employed in numerous broad fields, boundary value problems have garnered a lot of interest and are regarded as one of the key equations [1, 2] . Applied mathematics, numerous fields of physics, engineering, and chemistry all involve boundary value problems[1–5]. The Regge problem appears in the growth of quantum scattering when the support for interaction is constrained. The Sturm-Liouville equation on the semi-axis is the outcome of splitting the variables in the three-dimensional Schrödinger equation with radial symmetric potential, which is essentially the S-wave radial Schrödinger equation in physics.

$$-\psi''(\zeta,\lambda)+q(\zeta)\psi(\zeta,\lambda)=\lambda^2\psi(\zeta,\lambda). \ldots (1.1)$$

The above equation that can be translated into fractional order with T.Regge problem was first worked on by Karwan and Hozan in their paper [2, 6, 7].In this paper, we examine solutions to fractional boundary value problems. The interval on the half-axis R+ is supported compactly by the investigations of the Schrodinger operator with potential this problem manifests as

$$-{}_0^C D_\eta^\alpha\psi(\eta)+q(\eta)\psi(\eta)=\lambda^4 p(\eta)\psi(\eta), \qquad \eta\in[0,a],\ 3<\alpha\le 4 \ \ldots (1.2)$$

$$\psi(0)=\beta_1,\ \psi'(0)=\beta_2,\ \psi(a)=\beta_3\ ,\ \psi'(a)=\beta_4\ ..(1.3)$$

Where $\beta_1,\beta_2,\beta_3,\beta_4$ are constants and $q(\eta),p(\eta)\in L_+[0,a], where$

$L_+[0,a]=\{f(\eta):\int_a^b|f(\eta)|d\eta<\infty\}\ and\ 0<\ m\le f(\eta)\le M<\infty\ , and\ \alpha\in(3,4]\ , and\ \lambda$ is a spectral parameter.

This fractional boundary value problem, along with other fractional order calculus boundary value problems, is a helpful tool for comprehending the memory and inherited characteristics of various materials and processes.[8, 9] . Numerous scientific and technological domains, including as biology, chemistry, fluid mechanics, acoustics, viscoelasticity, anomalous diffusion, and control theory, can make use of it. In these instances, a family of separately presented integro-differential equations was solved using fractional differential equations. [7], [8]. The existence and uniqueness theorems for fractional ordinary differential equations were introduced. [3, 10].

and useful in many science ,physics and chemistry , engineering[5, 11]. There have previously been presented a number of analytical or fractional differential equations may be solved numerically in a variety of ways, including [9, 12, 13, 14].

2. Preliminaries

In this section, we provide several definitions, lemmas, and theorems that are fundamental to our theorems.

Definition 2.1 [10] The Gamma function is defined by the integral formula

$$\Gamma(l)=\int_0^\infty u^{l-1}e^{-u}\,du \quad \text{where } Re(l)>0.$$

Definition 2.2 [15] **(Fractional Integral)** a local integrable function $g(z)$, and for every $\gamma>0$ the right FI of order α is defined:

$${}_aI_z^\gamma g(z)=\int_a^z\frac{(z-s)^{\gamma-1}}{\Gamma(\gamma)}g(s)ds, \qquad -\infty\le a<z<\infty\ .$$

Definition 2.3 [8, 11] **(Fractional Derivative)** The derivative for order α for every $\alpha\in R$, and $m=[\alpha]$ the Riemann-Liouville is defined as follows:

$${}_aD_\zeta^\alpha g(z)=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{d\zeta^m}\int_a^\zeta(u-s)^{m-\alpha-1}g(s)ds\ .$$

Definition 2.4 [10, 11] Let $\alpha>0$, $m=[\alpha]$. The Caputo derivative of order $\alpha\in R$ and $f(u)$ be m −times differentiabl function, $u>a$ is defined as

$${}_a^C D_u^\alpha h(u)=\frac{1}{\Gamma(m-\alpha)}\int_a^u(u-s)^{m-\alpha-1}\left(\frac{d}{ds}\right)^m h(s)ds \quad \text{Or}$$

$${}_a^C D_u^\alpha h(u)=\frac{1}{\Gamma(m-\alpha)}\int_a^u\frac{h^m(s)}{(u-s)^{\alpha-m+1}}ds\ .$$

Remark 1: [2, 8, 11] The fractional differential and integration operators are linear.

Let $f(\eta),g(\eta)$ be two functions such that both ${}^C D_a^\alpha f(\zeta),\ {}^C D_a^\alpha g(\zeta)$ exist for $\alpha\in[m-1,m)$ and $a,b\in\mathbb{C}$. Then

$${}^C D_a^\alpha(af(\zeta)+bg(\zeta))=a\,{}^C D_a^\alpha f(\zeta)+b\,{}^C D_a^\alpha g(\zeta)$$

The Caputo operator of order's integration and differentiation has the following relationships:

- Caputo derivative of the fractional integral is ${}^C D_a^\alpha(I_a^\alpha f(u))=f(u)$.
- Fractional integral of the Caputo derivative is $I_a^\alpha({}^C D_a^\alpha f(u))=f(u)-\sum_{p=0}^{m-1}\frac{(u-a)^p}{p!}f^{(p)}(a)$.

From the above we got ${}^C D_a^\alpha(I_a^\alpha f(u))\ne I_a^\alpha({}^C D_a^\alpha f(u))$.

Remark 2: We have the following characteristics and definitions: The fractional integral of the two is equivalent, however unlike the fractional derivative of Caputo, the fractional integral of the derivatives order α of Caputo and Riemann-Liouville is linked). [10, 15]

Let $m\in\mathbb{N},\alpha\in[m-1,m)$. And let $f(u)$ be a function such that ${}^c D_a^\alpha f(u)$ and $D_a^\alpha f(u)$ exist. The (R-L) and Caputo derivatives are related in the following ways:

$${}^c D_a^\alpha f(u)=D_a^\alpha f(u)-\sum_{p=0}^{m-1}\frac{(u-a)^{p-\alpha}}{\Gamma(k+1-\alpha)}f^{(p)}(a).$$

3. Methodology

Definition 3.1 [16]

According to the following norm, the vector space $C[e,d]$ among the continuous complex-valued functions defined on a closed interval$[e,d]$ is a Banach space. $\|v\|_{C[e,d]}=max_{\eta\in[e,d]}|v(\eta)|\ \ ,v\in C[e,d]$

Definition 3.2 [16]

Let $H\ and\ S$ be two normed spaces and $T:H\to S$ the operator T it is said to be bounded if for positive number z such that$\|T\eta\|\le z\|\eta\|,\eta\in H.$

Lemma 3.3: Let $\psi(\eta)\in C(0,a]$ consequently, the boundary value problem's resolution (1.2)-(1.3) is

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$\psi(\eta)=\frac{1}{\Gamma(\alpha)}\int_0^\eta(\eta-\zeta)^{\alpha-1}\big(q(\zeta)-\lambda^4 p(\zeta)\big)\psi(\zeta)d\zeta-\frac{\eta^3}{a^2\Gamma(\alpha)}\int_0^a(a-\zeta)^{\alpha-2}\left(\alpha-1+a+\frac{2}{a}-\zeta\right)\big(q(\zeta)-\lambda^4 p(\zeta)\big)\psi(\zeta)d\zeta+\beta_1\left(1-\frac{3\eta^2}{a^2}+\frac{2\eta^3}{a^3}\right)+\beta_2\left(\eta-\frac{2\eta^2}{a}+\frac{\eta^3}{a^2}\right)+\beta_3\frac{(3a-1)}{a^3}\eta^3+\beta_4\frac{(\eta-a)}{a^2}\eta^2.$

Where $\beta_1,\beta_2,\beta_3,\beta_4$ are constants.

Proof: Equations 1.2 and 1.3 provide the Fractional Boundary Value Problem that we have.

$-{}_0^C D_z^\alpha k(z)+q(z).k(z)=\lambda^4 p(z).k(z)\,,\ \alpha\in(3,4]$

${}_0^C D_z^\alpha k(z)=q(z)k(z)-\lambda^4 p(z)k(z)$

Since $3<\alpha\le 4$ and b remark 1 we get

$I^\alpha\ {}_0^C D_z^\alpha k(z)=\psi(\eta)-\sum_{n=0}^{3}\frac{z^n}{n!}\psi^n(0)$

$I^\alpha\ {}_0^C D_z^\alpha k(z)=k(z)-k(0)-k'(0)z-\frac{z^2}{2}k''(0)-\frac{z^3}{3!}k'''(0)$

$I^\alpha\ {}_0^C D_z^\alpha k(z)=k(z)-\beta_1-\beta_2 z-\frac{z^2}{2}k''(0)-\frac{z^3}{3!}k'''(0)$

Now

$k(z)-\beta_1-\beta_2 z-\frac{z^2}{2}k''(0)-\frac{z^3}{3!}k'''(0)=I^\alpha\big(q(z)k(z)-\lambda^4 p(z)k(z)\big)$

$k(z)=I^\alpha\big(q(z)k(z)-\lambda^4 p(z)k(z)\big)+\beta_1+\beta_2 z+\frac{z^2}{2}k''(0)+\frac{z^3}{3!}k'''(0)$

Now to find $k''(0)$ and $k'''(0)$

$k(z)=\frac{1}{\Gamma(\alpha)}\int_0^z(z-\zeta)^{\alpha-1}\big(q(\zeta)-\lambda^4 p(\zeta)\big)k(\zeta)d\zeta+\beta_1+\beta_2 z+\frac{z^2}{2}k''(0)+\frac{z^3}{3!}k'''(0)$

Putting $z=a$ we get

$k(a)=\frac{1}{\Gamma(\alpha)}\int_0^a(a-\zeta)^{\alpha-1}\big(q(\zeta)-\lambda^4 p(\zeta)\big)k(\zeta)d\zeta+\beta_1+a\beta_2+\frac{a^2}{2}k''(0)+\frac{a^3}{3!}k'''(0)$

And

$k'(z)=\frac{\alpha-1}{\Gamma(\alpha)}\int_0^z(z-\zeta)^{\alpha-2}\big(q(\zeta)-\lambda^4 p(\zeta)\big)\psi(\zeta)d\zeta+\beta_2+z\psi''(0)+\frac{z^2}{2}\psi'''(0)$

Putting $\eta=a$ in $k'(z)$ we get

$k'(a)=\frac{\alpha-1}{\Gamma(\alpha)}\int_0^a(a-\zeta)^{\alpha-2}\big(q(\zeta)-\lambda^4 p(\zeta)\big)k(\zeta)d\zeta+\beta_2+ak''(0)+\frac{a^2}{2}k'''(0)$

From boundary conditions $\psi(a)=\beta_3\ ,\psi'(a)=\beta_4$

So

$\frac{a^2}{2}k''(0)+\frac{a^3}{3!}k'''(0)=\beta_3-\beta_1-a\beta_2-\frac{1}{\Gamma(\alpha)}\int_0^a(a-\zeta)^{\alpha-1}\big(q(\zeta)-\lambda^4 p(\zeta)\big)k(\zeta)d\zeta$

And $ak''(0)+\frac{a^2}{2}k'''(0)=\beta_4-\beta_2-\frac{\alpha-1}{\Gamma(\alpha)}\int_0^a(a-\zeta)^{\alpha-2}\big(q(\zeta)-\lambda^4 p(\zeta)\big)k(\zeta)d\zeta$

Let $A=\beta_3-\beta_1-a\beta_2-\frac{1}{\Gamma(\alpha)}\int_0^a(a-\zeta)^{\alpha-1}\big(q(\zeta)-\lambda^4 p(\zeta)\big)k(\zeta)d\zeta$

And $B=\beta_4-\beta_2-\frac{\alpha-1}{\Gamma(\alpha)}\int_0^a(a-\zeta)^{\alpha-2}\big(q(\zeta)-\lambda^4 p(\zeta)\big)k(\zeta)d\zeta$

Now $\frac{a^2}{2}k''(0)+\frac{a^3}{3!}k'''(0)=A$ .... (1.4)

$ak''(0)+\frac{a^2}{2}k'''(0)=B$ ... (1.5)

Solve equations 1.4 and 1.5 together to find $k''(0)$ and $k'''(0)$ we get

$k''(0)=\frac{6}{a^2}A-\frac{2}{a}B$ and $k'''(0)=\frac{6}{a^2}B-\frac{12}{a^3}A$

Put $k''(0)$ and $k'''(0)$ in above we have

$k(z)=\frac{1}{\Gamma(\alpha)}\int_0^z(z-\zeta)^{\alpha-1}\big(q(\zeta)-\lambda^4 p(\zeta)\big)k(\zeta)d\zeta+\beta_1+\beta_2 z+\frac{z^2}{2}\left(\frac{6}{a^2}A-\frac{2}{a}B\right)+\frac{z^3}{3!}\left(\frac{6}{a^2}B-\frac{12}{a^3}A\right)$

$k(z)=\frac{1}{\Gamma(\alpha)}\int_0^z(z-\zeta)^{\alpha-1}\big(q(\zeta)-\lambda^4 p(\zeta)\big)k(\zeta)d\zeta+\beta_1+\beta_2 z+z^2\frac{3}{a^2}A-z^2\frac{1}{a}B+z^3\frac{1}{a^2}B-z^3\frac{2}{a^3}A.$

$k(z)=\frac{1}{\Gamma(\alpha)}\int_0^z(z-\zeta)^{\alpha-1}\big(q(\zeta)-\lambda^4 p(\zeta)\big)k(\zeta)d\zeta-\frac{z^3}{a^2\Gamma(\alpha)}\int_0^a(a-\zeta)^{\alpha-2}\left(\alpha-1+a+\frac{2}{a}-\zeta\right)\big(q(\zeta)-\lambda^4 p(\zeta)\big)k(\zeta)d\zeta+\beta_1\left(1-\frac{3z^2}{a^2}+\frac{2z^3}{a^3}\right)+\beta_2\left(z-\frac{2z^2}{a}+\frac{z^3}{a^2}\right)+\beta_3\frac{(3a-1)}{a^3}z^3+\beta_4\frac{(z-a)}{a^2}z^2.$

As defined by equations 1.2 and 1.3, the fractional operator associated with the fractional boundary value issue is as follows:

$T:C[0,a]\to C[0,a]$ Is $Tk(z)=\frac{1}{\Gamma(\alpha)}\int_0^z(z-\zeta)^{\alpha-1}\big(q(\zeta)-\lambda^4 p(\zeta)\big)k(\zeta)d\zeta-\frac{z^3}{a^2\Gamma(\alpha)}\int_0^a(a-\zeta)^{\alpha-2}\left(\alpha-1+a+\frac{2}{a}-\zeta\right)\big(q(\zeta)-\lambda^4 p(\zeta)\big)k(\zeta)d\zeta+\beta_1\left(1-\frac{3z^2}{a^2}+\frac{2z^3}{a^3}\right)+\beta_2\left(z-\frac{2z^2}{a}+\frac{z^3}{a^2}\right)+\beta_3\frac{(3a-1)}{a^3}z^3+\beta_4\frac{(z-a)}{a^2}z^2.$

Lemma 3.4: if $|\beta_3|\le\|k\|$ then the operator T is bounded.

Proof: Define the operator $T:C[0,a]\to C[0,a]$ and let $k\in C[0,a]\,,\zeta\in[0,a]$

We will show that $\|T\psi\|\le l\|k\|\,,\ l\in R^+$,

$|Tk(z)|=\left|\frac{1}{\Gamma(\alpha)}\int_0^z(z-\zeta)^{\alpha-1}\big(q(\zeta)-\lambda^4 p(\zeta)\big)\psi(\zeta)d\zeta-\frac{z^3}{a^2\Gamma(\alpha)}\int_0^a(a-\zeta)^{\alpha-2}\left(\alpha-1+a+\frac{2}{a}-\zeta\right)\big(q(\zeta)-\lambda^4 p(\zeta)\big)k(\zeta)d\zeta+\beta_1\left(1-\frac{3z^2}{a^2}+\frac{2z^3}{a^3}\right)+\beta_2\left(z-\frac{2z^2}{a}+\frac{\eta^3}{a^2}\right)+\beta_3\frac{(3a-1)}{a^3}z^3+\beta_4\frac{(z-a)}{a^2}z^2\right|.$

$\frac{1}{\Gamma(\alpha)}\int_0^z|(z-\zeta)^{\alpha-1}|(|q(\zeta)|+|\lambda^4||p(\zeta)|)|z(\zeta)|d\zeta+\frac{z^3}{a^2\Gamma(\alpha)}\int_0^a(a-\zeta)^{\alpha-2}\left|\alpha-1+a+\frac{2}{a}-\zeta\right|(|q(\zeta)|+|\lambda^4||p(\zeta)|)|z(\zeta)|d\zeta+|\beta_1|\left|1-\frac{3z^2}{a^2}+\frac{2z^3}{a^3}\right|+|\beta_2|\left|z-\frac{2z^2}{a}+\frac{z^3}{a^2}\right|+|\beta_3|\frac{|3a-1|}{a^3}z^3+\beta_4\frac{|z-a|}{a^2}z^2$

$\le\frac{M(1+|\lambda^4|)}{\Gamma(\alpha)}\|k\|\int_0^a(a-\zeta)^{\alpha-1}d\zeta+\frac{aM(1+|\lambda^4|)}{\Gamma(\alpha)}\|k\|\int_0^a(a-\zeta)^{\alpha-2}\left(\alpha-1+a+\frac{2}{a}-\zeta\right)d\zeta+|\beta_3||3a-1|$

$\le\frac{a^\alpha M(1+|\lambda^4|)}{\alpha\Gamma(\alpha)}\|k\|+\frac{aM(1+|\lambda^4|)}{\Gamma(\alpha)}\|k\|\left(\frac{a^{\alpha-1}}{\alpha-1}\left(\alpha-1+a+\frac{2}{a}\right)-\frac{a^\alpha}{\alpha(\alpha-1)}\right)+\|k\||3a-1|$

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$\leq l\|k\|$, where $l = \frac{a^{\alpha}M(1+|\lambda^4|)}{\alpha\Gamma(\alpha)} + \frac{aM(1+|\lambda^4|)}{\Gamma(\alpha)}\left(\frac{a^{\alpha-1}}{\alpha-1}\left(\alpha-1+a+\frac{2}{a}\right) - \frac{a^{\alpha}}{\alpha(\alpha-1)}\right) + |3a-1|$

So the operator T is bounded.

Definition 3.5: [17] $for\ all\ \varepsilon > 0\ there\ exist\ \delta > 0\ , and\ for\ all\ \eta_1, \eta_2 \in D(f_n)$ a sequence function f_n is said to be equicontinuous if such that $|\eta_2 - \eta_1| < \delta$ then $|f_n(\eta_2) - f_n(\eta_1)| < \varepsilon$.

Lemma 3.6: The operator T is equicontinuous.

Proof: Define the operator $T: C[0,a] \to C[0,a]$ and let $\psi \in C[0,a]$, $\zeta \in [0,a]$

If

$for\ all\ \varepsilon > 0\ there\ exist\ \delta > 0\ , and\ for\ all\ \psi(\zeta_1), \psi(\zeta_2) \in D(T)$ such that $|\zeta_2 - \zeta_1| < \delta$ then $|T\psi(\zeta_2) - T\psi(\zeta_1)| < \varepsilon$

Suppose that $B_{\gamma} = \{\psi \in C[0,a], \|\psi\|_{C[0,a]} \leq \gamma\}, \gamma > 0$, let $0 \leq \zeta \leq \zeta_1 \leq \zeta_2 \leq a$

$$|T\psi(\zeta_2) - T\psi(\zeta_1)| = \left|\frac{1}{\Gamma(\alpha)}\int_0^{\zeta_2}(\zeta_2-\zeta)^{\alpha-1}\left(q(\zeta) - \lambda^4 p(\zeta)\right)\psi(\zeta)d\zeta - \frac{1}{\Gamma(\alpha)}\int_0^{\zeta_1}(\zeta_1-\zeta)^{\alpha-1}\left(q(\zeta) - \lambda^4 p(\zeta)\right)\psi(\zeta)d\zeta + \frac{({\zeta_1}^3 - {\zeta_2}^3)}{a^2\Gamma(\alpha)}\int_0^a (a-\zeta)^{\alpha-2}\left(\alpha - 1 + a + \frac{2}{a} - \zeta\right)\left(q(\zeta) - \lambda^4 p(\zeta)\right)\psi(\zeta)d\zeta + \beta_1\left(\frac{3}{a^2}({\zeta_1}^2 - {\zeta_2}^2) + \frac{2}{a^3}({\zeta_2}^3 - {\zeta_1}^3)\right) + \beta_2\left((\zeta_2 - \zeta_1) + \frac{2}{a}({\zeta_1}^2 - {\zeta_2}^2) + \frac{1}{a^2}({\zeta_2}^3 - {\zeta_1}^3)\right) + \beta_3\frac{(3a-1)}{a^3}({\zeta_2}^3 - {\zeta_1}^3) + \frac{\beta_4}{a^2}\left((\zeta_2 - a){\zeta_2}^2 - (\zeta_1 - a){\zeta_1}^2\right)\right|.$$

Since $q(\eta), p(\eta) \in L^{+}[0,a]$ so $max_{\eta\in[0,a]}q(\eta) = M = max_{\eta\in[0,a]}p(\eta)$

And $|a-b| = |b-a|\ and max_{\eta\in[0,a]}|\psi(\eta)| = \|\psi\|$ we get

$$|T\psi(\zeta_2) - T\psi(\zeta_1)| \leq \frac{M(1+|\lambda^4|)}{\Gamma(\alpha)}\|\psi\|\left|\int_0^{\zeta_2}(\zeta_2-\zeta)^{\alpha-1}d\zeta - \int_0^{\zeta_1}(\zeta_1-\zeta)^{\alpha-1}d\zeta\right| + |\zeta_2 - \zeta_1|\left|{\zeta_1}^2 + \zeta_2\zeta_1 + {\zeta_2}^2\right|\|\psi\|\frac{M(1+|\lambda^4|)}{a^2\Gamma(\alpha)}\left|\int_0^a (a-\zeta)^{\alpha-2}\left(\alpha - 1 + a + \frac{2}{a} - \zeta\right)d\zeta\right| + \beta_1\left(\frac{3}{a^2}|\zeta_2 - \zeta_1||\zeta_2 + \zeta_1| + \frac{2}{a^3}|\zeta_2 - \zeta_1|\left|{\zeta_1}^2 + \zeta_2\zeta_1 + {\zeta_2}^2\right|\right) + \beta_2\left(|\zeta_2 - \zeta_1| + \frac{2}{a}|\zeta_2 - \zeta_1||\zeta_2 + \zeta_1| + \frac{1}{a^2}|\zeta_2 - \zeta_1|\left|{\zeta_1}^2 + \zeta_2\zeta_1 + {\zeta_2}^2\right|\right) + \beta_3\frac{(3a-1)}{a^3}|\zeta_2 - \zeta_1|\left|{\zeta_1}^2 + \zeta_2\zeta_1 + {\zeta_2}^2\right| + \frac{\beta_4}{a^2}\left(|\zeta_2 - \zeta_1|\left|{\zeta_1}^2 + \zeta_2\zeta_1 + {\zeta_2}^2\right| + a|\zeta_2 - \zeta_1||\zeta_2 + \zeta_1|\right).$$

And we have $|\zeta_2 - \zeta_1| \leq \delta$ and $\zeta_1, \zeta_2 \in [0,a]$ so implies that

$$|T\psi(\zeta_2) - T\psi(\zeta_1)| \leq \frac{M\gamma(1+|\lambda^4|)}{\Gamma(\alpha+1)}\delta + 3\delta a^2\frac{M\gamma(1+|\lambda^4|)}{a^2\Gamma(\alpha)}\left|\int_0^a (a-\zeta)^{\alpha-2}\left(\alpha - 1 + a + \frac{2}{a} - \zeta\right)d\zeta\right| + \beta_1\delta\left(\frac{12}{a}\right) + 8\beta_2\delta + \beta_3\delta\frac{3(3a-1)}{a} + 5\delta\beta_4.$$

Take

$$\delta = \frac{\epsilon}{\frac{M\gamma(1+|\lambda^4|)}{\Gamma(\alpha+1)} + 3a^2\frac{M\gamma(1+|\lambda^4|)}{a^2\Gamma(\alpha)}\left|\int_0^a (a-\zeta)^{\alpha-2}\left(\alpha-1+a+\frac{2}{a}-\zeta\right)d\zeta\right| + \beta_1\left(\frac{12}{a}\right) + 8\beta_2 + \beta_3\frac{3(3a-1)}{a} + 5\beta_4}$$

We get $|T\psi(\zeta_2) - T\psi(\zeta_1)| \leq \epsilon$

Therefore T is equicontiuous.

Definition 3.7 (Contraction) [18] A mapping $T: E \longrightarrow E$ is said to be contraction on E (E,d) if there is a positive real number $h < 1$ such that for any $\eta, \psi \in E$, then is a full metric space.

$d(T\eta, T\psi) \leq hd(\eta,\psi)$.

According to geometry, any points x and y have images that are closer together than those locations x and y that are more accurately spaced apart. Additionally, the ratio of $d(T\eta, T\psi)/d(\eta,\psi)$ cannot go above a limit h that is unmistakably less than 1.

Theorem 3.8: (Banach Fixed Point Theorem) [7, 19]

If $T: S \to S$ if T has a distinct fixed point in S and C is a contraction operator defined on S.

Theorem 3.9: Existence and Uniqueness Theorem

If the condition is true, the Fractional Boundary Value Problem (FBVP) presented in sections 2.2 and 2.3 has a unique solution if

$D < 1$; Where $D = \frac{a^{\alpha-1}M(1+|\lambda^4|)(\alpha a^2 + \alpha a^2 - a^2 - a + 2\alpha)}{(\alpha-1)\Gamma(\alpha+1)}$

proof: Define the operator T as $T: C[0,a] \to C[0,a]$ is

$$T\psi(\eta) = \frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\zeta)^{\alpha-1}\left(q(\zeta) - \lambda^4 p(\zeta)\right)\psi(\zeta)d\zeta - \frac{\eta^3}{a^2\Gamma(\alpha)}\int_0^a (a-\zeta)^{\alpha-2}\left(\alpha - 1 + a + \frac{2}{a} - \zeta\right)\left(q(\zeta) - \lambda^4 p(\zeta)\right)\psi(\zeta)d\zeta + \beta_1\left(1 - \frac{3\eta^2}{a^2} + \frac{2\eta^3}{a^3}\right) + \beta_2\left(\eta - \frac{2\eta^2}{a} + \frac{\eta^3}{a^2}\right) + \beta_3\frac{(3a-1)}{a^3}\eta^3 + \beta_4\frac{(\eta-a)}{a^2}\eta^2.$$

Let $u, v \in C[0,a]$ So we have

$$|Tu(\eta) - Tv(\eta)| = \left|\frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\zeta)^{\alpha-1}\left(q(\zeta) - \lambda^4 p(\zeta)\right)u(\zeta)d\zeta - \frac{\eta^3}{a^2\Gamma(\alpha)}\int_0^a (a-\zeta)^{\alpha-2}\left(\alpha - 1 + a + \frac{2}{a} - \zeta\right)\left(q(\zeta) - \lambda^4 p(\zeta)\right)u(\zeta)d\zeta + \beta_1\left(1 - \frac{3\eta^2}{a^2} + \frac{2\eta^3}{a^3}\right) + \beta_2\left(\eta - \frac{2\eta^2}{a} + \frac{\eta^3}{a^2}\right) + \beta_3\frac{(3a-1)}{a^3}\eta^3 + \beta_4\frac{(\eta-a)}{a^2}\eta^2 - \frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\zeta)^{\alpha-1}\left(q(\zeta) - \lambda^4 p(\zeta)\right)v(\zeta)d\zeta + \frac{\eta^3}{a^2\Gamma(\alpha)}\int_0^a (a-\zeta)^{\alpha-2}\left(\alpha - 1 + a + \frac{2}{a} - \zeta\right)\left(q(\zeta) - \lambda^4 p(\zeta)\right)v(\zeta)d\zeta - \beta_1\left(1 - \frac{3\eta^2}{a^2} + \frac{2\eta^3}{a^3}\right) - \beta_2\left(\eta - \frac{2\eta^2}{a} + \frac{\eta^3}{a^2}\right) - \beta_3\frac{(3a-1)}{a^3}\eta^3 - \beta_4\frac{(\eta-a)}{a^2}\eta^2\right|.$$

$$\leq \left|\frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\zeta)^{\alpha-1}\left(q(\zeta) - \lambda^4 p(\zeta)\right)|u(\zeta) - v(\zeta)|d\zeta\right| + \left|\frac{\eta^3}{a^2\Gamma(\alpha)}\int_0^a (a-\zeta)^{\alpha-2}\left(\alpha - 1 + a + \frac{2}{a} - \zeta\right)\left(q(\zeta) - \lambda^4 p(\zeta)\right)|v(\zeta) - u(\zeta)|d\zeta\right|.$$

Taking $max_{\zeta\in[0,a]}$ we get

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$\leq \frac{M(1+|\lambda^4|)}{\Gamma(\alpha)}\|u-v\|\left|\int_0^a (a-\zeta)^{\alpha-1}d\zeta\right| + \left|\frac{aM(1+|\lambda^4|)}{\Gamma(\alpha)}\|u-v\|\int_0^a (a-\zeta)^{\alpha-2}\left(\alpha-1+a+\frac{2}{a}-\zeta\right)d\zeta\right|$, $since\ q(\eta), p(\eta) \in L_+[0,a]$.

$\leq \frac{a^\alpha M(1+|\lambda^4|)}{\alpha\Gamma(\alpha)}\|u-v\| + \frac{aM(1+|\lambda^4|)}{\Gamma(\alpha)}\|u-v\|\left(\frac{a^{\alpha-1}}{\alpha-1}\left(\alpha-1+a+\frac{2}{a}\right)-\frac{a^\alpha}{\alpha(\alpha-1)}\right)$

$= \frac{a^\alpha M(1+|\lambda^4|)}{\Gamma(\alpha+1)}\|u-v\|\left(\frac{\alpha a^2+\alpha a^2-a^2-a+2\alpha}{a(\alpha-1)}\right) = \frac{a^{\alpha-1}M(1+|\lambda^4|)(\alpha a^2+\alpha a^2-a^2-a+2\alpha)}{(\alpha-1)\Gamma(\alpha+1)}\|u-v\|$

$= D\|u-v\|$, Where $D = \frac{a^{\alpha-1}M(1+|\lambda^4|)(\alpha a^2+\alpha a^2-a^2-a+2\alpha)}{(\alpha-1)\Gamma(\alpha+1)}$

Now $|Tu(\eta)-Tv(\eta)| \leq D\|u-v\|_{C[0,a]}$
$\rightarrow$
$max_{\eta\in[0,a]}|Tu(\eta)-Tv(\eta)| \leq max_{\eta\in[0,a]}D\|u(\eta)-v(\eta)\|_{C[0,a]}$

Therefore $\|Tu-Tv\| \leq D\|u-v\|_{C[0,a]}$

Since $D<1$ then T is a contraction operator. The singular fixed point for T is determined by the Banach Fixed Point Theorem and is supplied by equations 1.2 and 1.3 as the unique solution to the fractional boundary value problem. ■

Definition 3.10: [18] An operator $T: H \rightarrow H$ is said to be compact if for each bounded sequence $\emptyset_n \in H$, $T(\emptyset_n)$ has a convergent subsequence .

Theorem 3.11: [18] **(Arzela Theorem)** There is a convergent subsequence for any bounded equicontinuous function. Every operator that is bounded and equicontinuous is compact.

Theorem 3.12: [19] **(Leray-Schauder Fixed Point Theorem)**

Let Y be a Banach space and let $A: Y \longrightarrow Y$ being a small operator Let's say the set
$N = \{\psi \in Y \mid \psi = \mu T\psi\ for\ some\ "\ \mu \in [0,1]\}$Is bounded, and then A has at least one fixed point.

We shall present a few theorems to demonstrate the existence and uniqueness theorem for problems 1.2 and 1.3 of fractional order.

Theorem 3.13: Suppose that there exist real number G > 0, such that $\beta_3 \leq \|\psi\|_{C[0,a]} \leq$ G , then the fractional boundary value problem given by equations (1.2)-(1.3) has at least one solution.

Proof: let T be the operator $T: C[0,a] \rightarrow C[0,a]$ and let $\psi \in C[0,a]$, $\eta \in [0,a]$

By (Lemma 1) we can define the operator T related to fractional differential equation as

$$T\psi(\eta) = \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\zeta)^{\alpha-1}\left(q(\zeta)-\lambda^4 p(\zeta)\right)\psi(\zeta)d\zeta - \frac{\eta^3}{a^2\Gamma(\alpha)}\int_0^a (a-\zeta)^{\alpha-2}\left(\alpha-1+a+\frac{2}{a}-\zeta\right)\left(q(\zeta)-\lambda^4 p(\zeta)\right)\psi(\zeta)d\zeta + \beta_1\left(1-\frac{3\eta^2}{a^2}+\frac{2\eta^3}{a^3}\right) + \beta_2\left(\eta-\frac{2\eta^2}{a}+\frac{\eta^3}{a^2}\right) + \beta_3\frac{(3a-1)}{a^3}\eta^3 + \beta_4\frac{(\eta-a)}{a^2}\eta^2.$$

Let $L = \frac{a^\alpha M(1+|\lambda^4|)}{\Gamma(\alpha+1)} + \frac{a^\alpha M(1+|\lambda^4|)}{\alpha-1\Gamma(\alpha+1)}\left(\alpha^2-\alpha+a\alpha+\frac{2\alpha}{a}-a\right)+(3a-1)$, $and\ R = GL$

And let $M = \{\psi \in C[0,a] \mid \psi = \mu T\psi\ for\ some\ \ \mu \in [0,1]\}$

We must prove that M is a bounded set, to do this suppose that $\psi \in M$ if $\psi = 0$ then $M = \{0\}$ is bounded and by using Leray-Schaude fixed point theorem operator T has a fixed point, this fixed point is a solution of the integral equation related to fractional differential equation given by 1.2 and 1.3.

If $\psi \in M$ and $\psi \neq 0$, so we have

$$|\psi(\eta)| = |\mu T\psi(\eta)| = \mu\left|\frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\zeta)^{\alpha-1}\left(q(\zeta)-\lambda^4 p(\zeta)\right)\psi(\zeta)d\zeta - \frac{\eta^3}{a^2\Gamma(\alpha)}\int_0^a (a-\zeta)^{\alpha-2}\left(\alpha-1+a+\frac{2}{a}-\zeta\right)\left(q(\zeta)-\lambda^4 p(\zeta)\right)\psi(\zeta)d\zeta + \beta_1\left(1-\frac{3\eta^2}{a^2}+\frac{2\eta^3}{a^3}\right) + \beta_2\left(\eta-\frac{2\eta^2}{a}+\frac{\eta^3}{a^2}\right) + \beta_3\frac{(3a-1)}{a^3}\eta^3 + \beta_4\frac{(\eta-a)}{a^2}\eta^2.\right|,$$

$$\leq \left|\frac{1}{\Gamma(\alpha)}\int_0^a (a-\zeta)^{\alpha-1}\left(q(\zeta)+\lambda^4 p(\zeta)\right)\psi(\zeta)d\zeta + \frac{a}{\Gamma(\alpha)}\int_0^a (a-\zeta)^{\alpha-2}\left(\alpha-1+a+\frac{2}{a}-\zeta\right)\left(q(\zeta)+\lambda^4 p(\zeta)\right)\psi(\zeta)d\zeta + \beta_3(3a-1)\right|,$$

$$\leq \left|\frac{M(1+|\lambda^4|)}{\Gamma(\alpha)}\|\psi\|\int_0^a (a-\zeta)^{\alpha-1}d\zeta + \frac{aM(1+|\lambda^4|)}{\Gamma(\alpha)}\|\psi\|\int_0^a (a-\zeta)^{\alpha-2}\left(\alpha-1+a+\frac{2}{a}-\zeta\right)d\zeta + \|\psi\|(3a-1)\right|,$$

$$\leq \left|\frac{a^\alpha M(1+|\lambda^4|)}{\alpha\Gamma(\alpha)}\|\psi\| + \frac{aM(1+|\lambda^4|)}{\Gamma(\alpha)}\|\psi\|\left(\frac{a^{\alpha-1}}{\alpha-1}\left(\alpha-1+a+\frac{2}{a}\right)-\frac{a^\alpha}{\alpha(\alpha-1)}\right)+\|\psi\|(3a-1)\right|.$$

$$= \|\psi\|\left[\frac{a^\alpha M(1+|\lambda^4|)}{\Gamma(\alpha+1)} + \frac{a^\alpha M(1+|\lambda^4|)}{\alpha-1\Gamma(\alpha+1)}\left(\alpha^2-\alpha+a\alpha+\frac{2\alpha}{a}-a\right)+(3a-1)\right],$$

$\leq GL = R$. Since $sup_{\eta\in[0,a]}|\psi(\eta)| = \|\psi\|_{C[0,a]} \leq R.$

This implies that M is a bounded set, and since T is a bounded and equicontinuous operator according to lemma 1.2 and 1.3, the operator T is compact and fully continuous according to the Arzela theorem. As a result, T has a fixed point according to the Leray-Schauder fixed point theorem, and this fixed point is the solution to the fractional boundary value problem stated by 1.2 and 1.3.

Theorem 3.14: (Schauder Fixed Point Theorem) [18, 20]

In a Banach space E, let B be a nonempty, convex, closed, and bounded set, and let $T: B \rightarrow B$ be a compact operator. When that happens, T has at least one fixed point in B .

Theorem 3.15: If there exist real number Z > 0, such that

$$Z = \frac{\beta_3(3a-1)(\alpha-1)\Gamma(\alpha+1)}{(\alpha-1)\Gamma(\alpha+1)-a^\alpha M(1+|\lambda^4|)\left((\alpha-1)+\left(\alpha^2-\alpha+a\alpha+\frac{2\alpha}{a}-a\right)\right)}$$

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Therefore there is at least one solution to the fractional boundary value issue presented by equations (1.2)–(1.3).

Proof: Define the operator $T: C[0,a] \to C[0,a]$ and let $\psi(\eta) \in C[0,a], \eta \in [0,a]$ as in theorem 1.4. Let $Z = \frac{\beta_3(3a-1)(\alpha-1)\Gamma(\alpha+1)}{(\alpha-1)\Gamma(\alpha+1)-a^\alpha M(1+|\lambda^4|)\left((\alpha-1)+\left(\alpha^2-\alpha+a\alpha+\frac{2\alpha}{a}-a\right)\right)}$

and assume

$B_Z = \{\psi \in C[0,a], \| \psi \|_{C[0,a]} \leq Z\}$.

The existence of B_z as a nonempty, closed, convex, and bounded subset of $C[0,a]$ may thus be verified with simplicity.

It's clear that B_Z is closed and bounded set, and convex set

Now to prove operator T on B_Z is bounded, let $\psi \in B_Z$, then

$$|T\psi(\eta)| = \left|\frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\zeta)^{\alpha-1}\left(q(\zeta)-\lambda^4 p(\zeta)\right)\psi(\zeta)d\zeta - \frac{\eta^3}{a^2\Gamma(\alpha)}\int_0^a (a-\zeta)^{\alpha-2}\left(\alpha-1+a+\frac{2}{a}-\zeta\right)\left(q(\zeta)-\lambda^4 p(\zeta)\right)\psi(\zeta)d\zeta + \beta_1\left(1-\frac{3\eta^2}{a^2}+\frac{2\eta^3}{a^3}\right) + \beta_2\left(\eta-\frac{2\eta^2}{a}+\frac{\eta^3}{a^2}\right) + \beta_3\frac{(3a-1)}{a^3}\eta^3 + \beta_4\frac{(\eta-a)}{a^2}\eta^2\right|.$$

$$\leq \left|\frac{a^\alpha M(1+|\lambda^4|)}{\Gamma(\alpha+1)}\|\psi\| + \frac{a^\alpha M(1+|\lambda^4|)}{(\alpha-1)\Gamma(\alpha+1)}\|\psi\|\left(\alpha^2-\alpha+a\alpha+\frac{2\alpha}{a}-a\right) + \beta_3(3a-1)\right|.$$

$$\leq \beta_3(3a-1) + \left(\frac{a^\alpha M(1+|\lambda^4|)}{\Gamma(\alpha+1)} + \frac{a^\alpha M(1+|\lambda^4|)}{(\alpha-1)\Gamma(\alpha+1)}\left(\alpha^2-\alpha+a\alpha+\frac{2\alpha}{a}-a\right)\right)Z.$$

$$\leq \beta_3(3a-1) + \left(\frac{a^\alpha M(1+|\lambda^4|)}{\Gamma(\alpha+1)} + \frac{a^\alpha M(1+|\lambda^4|)}{(\alpha-1)\Gamma(\alpha+1)}\left(\alpha^2-\alpha+a\alpha+\frac{2\alpha}{a}-a\right)\right)\frac{\beta_3(3a-1)(\alpha-1)\Gamma(\alpha+1)}{(\alpha-1)\Gamma(\alpha+1)-a^\alpha M(1+|\lambda^4|)\left((\alpha-1)+\left(\alpha^2-\alpha+a\alpha+\frac{2\alpha}{a}-a\right)\right)} \quad \text{... (1.6)}$$

$$= \frac{\beta_3(3a-1)(\alpha-1)\Gamma(\alpha+1)}{(\alpha-1)\Gamma(\alpha+1)-a^\alpha M(1+|\lambda^4|)\left((\alpha-1)+\left(\alpha^2-\alpha+a\alpha+\frac{2\alpha}{a}-a\right)\right)} = Z.$$

Therefore $\|T\psi(\eta)\| \leq Z$.

So we get $TB_Z \subseteq B_Z$, that means the operator T on B_Z is bounded, and clearly T is equicontinuous and bounded operator on $C[0,a]$, by Arzela Theorem operator is compact so by applying Schauder fixed point theorem, T has at least one solution.

4. Illustrative Examples

In this part, we'll provide a few examples.

Example 4.1 solve the fractional boundary value problem

$$-{}_0^C D_\eta^{\frac{10}{3}}\psi(\eta) + \frac{1}{2}\psi(\eta) = \lambda^4\frac{1}{2}\psi(\eta) \; ; \quad \eta \in [0,1]$$

$$\psi(0) = 0\,, \quad \psi'(0) = 0\;, \psi(1) = 1\,, \psi'(1) = 2$$

Solution: we have $M = \frac{1}{2}$ we'll use the Laplace transform method to solve this problem.

Now $-{}_0^C D_\eta^{\frac{10}{3}}\psi(\eta) + \frac{1}{2}\psi(\eta) = \frac{1}{2}\lambda^4\psi(\eta) \to$

${}_0^C D_\eta^{\frac{10}{3}}\psi(\eta) = \frac{1}{2}(1-\lambda^4)\psi(\eta)$

By using the Laplace transform and applying it to both sides, we demonstrate it.

$L\left\{{}_0^C D_\eta^{\frac{10}{3}}\psi(\eta)\right\} = L\{\frac{1}{2}(1-\lambda^4)\psi(\eta)\}$ By properties of Laplace transform we have

$$L\left\{{}_0^C D_\eta^{\frac{10}{3}}\psi(\eta)\right\} = \left\{s^{\frac{10}{3}}Y(s) - s^{\frac{7}{3}}\psi(0) - s^{\frac{4}{3}}\psi'(0) - s^{\frac{1}{3}}\psi''(0) - s^{\frac{-2}{3}}\psi'''(0)\right\}$$

Now $s^{\frac{10}{3}}Y(s) - s^{\frac{7}{3}}\psi(0) - s^{\frac{4}{3}}\psi'(0) - s^{\frac{1}{3}}\psi''(0) - s^{\frac{-2}{3}}\psi'''(0) = \frac{1}{2}(1-\lambda^4)Y(s)$

$s^{\frac{10}{3}}Y(s) - s^{\frac{1}{3}}A - s^{\frac{-2}{3}}B = \frac{1}{2}(1-\lambda^4)Y(s)$, where $A = \psi''(0)\,, B = \psi'''(0)$

$\left(s^{\frac{10}{3}} + \frac{1}{2}(\lambda^4-1)\right)Y(s) = s^{\frac{1}{3}}A + s^{\frac{-2}{3}}B$, such that $A, B \neq 0$

$$Y(s) = \frac{As^{\frac{1}{3}}}{s^{\frac{10}{3}}+\frac{1}{2}(\lambda^4-1)} + \frac{Bs^{\frac{-2}{3}}}{s^{\frac{10}{3}}+\frac{1}{2}(\lambda^4-1)}$$

Take Laplace inverse for both sides we get

$$L^{-1}(Y(s)) = L^{-1}\left(\frac{As^{\frac{1}{3}}}{s^{\frac{10}{3}}+\frac{1}{2}(\lambda^4-1)} + \frac{Bs^{\frac{-2}{3}}}{s^{\frac{10}{3}}+\frac{1}{2}(\lambda^4-1)}\right)$$

See [20]for inverse Laplace and related to Mittag-leffler we get

So

$$\psi(\eta) = A\eta^2 E_{\frac{10}{3},3}\left(\frac{1}{2}(1-\lambda^4)\eta^{\frac{10}{3}}\right) + B\eta^3 E_{\frac{10}{3},4}\left(\frac{1}{2}(1-\lambda^4)\eta^{\frac{10}{3}}\right) \;,$$

From the Mittag-Leffler definition, we can discover that λ .

The above fractional Boundary value Problem has solution as form

$$(\eta) = A\eta^2 E_{\frac{10}{3},3}\left(\frac{1}{2}(1-\lambda^4)\eta^{\frac{10}{3}}\right) + B\eta^3 E_{\frac{10}{3},4}\left(\frac{1}{2}(1-\lambda^4)\eta^{\frac{10}{3}}\right) \;, \quad \lambda \neq 1, -1, -i\,,$$

Example 4.2 consider the fractional boundary value problem

$$-{}_0^C D_\eta^{\frac{11}{3}}\psi(\eta) + 0.3\psi(\eta) = 0.3\lambda^4\psi(\eta); \quad \eta \in [0,1]$$

$$\psi(0) = 0\,, \quad \psi'(0) = 1\,, \quad \psi(1) = 0\,, \quad \psi'(1) = -1$$

Solution: now $-{}_0^C D_\eta^{\frac{11}{3}}\psi(\eta) + 0.3\psi(\eta) = 0.3\lambda^4\psi(\eta) \to {}_0^C D_\eta^{\frac{11}{3}}\psi(\eta) = 0.3(1-\lambda^4)\psi(\eta)$

See reference [15] page 55 we can see $D^\alpha\psi(\zeta) = h\psi(\zeta)$, *where:* $n-1 < \alpha < n$,

With the Boundary condition $\psi^{(k)}(0) = b_k$, $b_k \in R$, $k = 0,1,2,\dots,n-1$,

Has the solution: $\psi(\zeta) = \sum_{k=0}^{n-1} b_k\zeta^k E_{\alpha,k+1}(h\zeta^\alpha)$

We know $h = 0.3(1-\lambda^4)$, $3 < \alpha \leq 4$, $\psi(0) = b_0 = 0$, $\psi'(0) = b_1 = 1$ so the solution is

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$\psi(\eta) = \sum_{k=0}^{3} 0.3 b_k \eta^k E_{\frac{11}{3},k+1}\left(0.3(1-\lambda^4)\eta^{\frac{11}{3}}\right) = b_0 E_{\frac{11}{3},1}\left(0.3(1-\lambda^4)\eta^{\frac{11}{3}}\right) + b_1\eta^1 E_{\frac{11}{3},2}\left(0.3(1-\lambda^4)\eta^{\frac{11}{3}}\right) + b_2\eta^2 E_{\frac{11}{3},3}\left(0.3(1-\lambda^4)\eta^{\frac{11}{3}}\right) + b_3\eta^3 E_{\frac{11}{3},4}\left(0.3(1-\lambda^4)\eta^{\frac{11}{3}}\right) = \eta E_{\frac{5}{3},2}\left(0.3(1-\lambda^4)\eta^{\frac{5}{3}}\right) + b_2\eta^2 E_{\frac{11}{3},3}\left(0.3(1-\lambda^4)\eta^{\frac{11}{3}}\right) + b_3\eta^3 E_{\frac{11}{3},4}\left(0.3(1-\lambda^4)\eta^{\frac{11}{3}}\right).$

$\lambda \neq 1, -1, -i\, , b_2 = \psi''(0)\, , b_3 = \psi''(0)\, , b_2, b_3 \neq 0$

Conclusion

The Banach fixed point theorem (contraction mapping theorem) and the Schauder fixed point theorem have both been used to solve the fractional order boundary value issues 1.2 and 1.3, demonstrating their existence and uniqueness. The results demonstrate that the issue in sections 1.2 and 1.3 has just one potential resolution. The condition has been achieved using the operator we described for the issue.

Conflict of Interest: The authors affirm that they do not have any competing interests.

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