IASJ المجلات الاكاديمية العراقية

Tikrit Journal of Pure Science

ISSN: 1813 – 1662 (Print) --- E-ISSN: 2415 – 1726 (Online)

Journal Homepage: http://tjps.tu.edu.iq/index.php/j

Convergence solution for some Harmonic Stochastic Differential Equations with Application

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ARTICLE INFO.

Article history:

-Received: 18 / 4 / 2020

-Accepted: 30 / 6 / 2020

-Available online: / / 2020

Keywords: Stochastic differential equations, Error, Convergence, Euler-Maruyama method; Milstein method .

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ABSTRACT

The purpose of this paper is to provide an introduction to the theory, computation, and application of stochastic differential equations and also we study the exact and approximate solution for some harmonic stochastic differential equations , by using Ito integral formula and numerical approximation(the Euler-Maruyama method and the Milstein method) in order discuss the convergence accuracy of their solution. Also we proposed Intermediate points τ_k for the generalization to Ito integral formula and stratonovich formula. Milstein method is more accurate than Euler Maruyama method . By looking at the convergence rates of both methods , we find that Euler-Maruyama method is strongly convergent with $\gamma = 0.5$ and weakly convergent with $\gamma = 1$, whereas Milstein method is strongly and weakly convergent with $\gamma = 1$.

Introduction

In many application such as Ecology , Physics, Population Dynamic , Finance , Economic and many areas stochastic differential equation (SDE) play an important contributions[1], random event have provided a natural surrogate for deterministic function as mathematical characterization in the form of random equation such equation is called stochastic differential equation [2]. However, in numerous applications, the measured tracks do not in fact act empirical for systems intended by ordinary differential equation as predictable: the spotted state follows track prophesied by ordinary differential equation in one road or another, but also seem to be topic to random disturbances [3]. Consider the determine differential equation

$$\begin{cases} \dot{X}(t) = b\big(t, X(t)\big) \ ; \quad t > 0 \\ X(0) = X_0 \end{cases} \ ; \ \ldots.(1)$$

$b(\cdot)$ is any smooth function and $X_0$is any fixed point $X_0 \in \mathbb{R}^n$, Logically modified equation (1) to contain the potential of stochastic effects, so that

$$\{\dot{X}(t) = b\big(t, X(t)\big) + B(t, X(t))\xi(t) \ ; \ t > 0 \, . \ \ldots..(2)$$

Where $\xi(t)$ is white noise process(formal derivative of wiener process), .or, we can write (2) as

$$\left\{\frac{dX(t)}{dt} = b(t, X(t)) + B(t, X(t))\frac{dW(t)}{dt} \right. \ \ldots..(3)$$

Where $\frac{dW(t)}{dt} = \xi(t)$, $dW(t)$ indicate differential form of the Brownian motion, multiply by dt , We get:

$$\begin{cases} dX(t) = b\big(t, X(t)\big)dt + B(t, X(t))dW(t) \\ X(0) = X_0 \end{cases} \ \ldots..(4)$$

Where $b(.)$ and $B(.)$ are drift and deffnsion coefficient respectively .

Equation (4) is called stochastic differential equations(SDE) .

In (2012), A. Tocino and M. j. Senosiain [4], study the mean square stability analysis of numerical schemes for stochastic differential systems., In (2016), Arsalane Chouaib and Kamal Boukhetala [5] study the estimation of stochastic differential equation with Simulation of Diffusion Processes ., M. Mascaro Monserrat 2017 [6] studied how to obtain stochastic differential equations by using Ito Stochastic integrals . Simo Särkkä and Arno Solin in (2019) [7] studied applications stochastic differential equation . Kristina R. Dahl studied A numerical method for solving stochastic differential equations with noisy memory. Ayman M.M. & Cemil Tunc in (2019) [8]. find a criteria for asymptotically stability of the zero solution for a kind of three-order stochastic differential system with delays .

In this paper we study the convergence of some harmonic stochastic differential equations, in order to confirm the accuracy of convergence of the numerical methods with the exact solution for the proposed model .

Existence of a Unique of Solution [9] :

A solution to a random differential equation is called strong, if it is possible, with respect to Brownian motion *W(t)*, to construct a solution *X(t)* that is unique to that given structural motion. This means that the full process path is unique to a specific Brownian motion. Hence strong singularity is also called the uniqueness of the wise path.

The strong uniqueness of a solution to stochastic differential equation (SDE) of the general form

$dX(t) = b(t, X(t))dt + B(t, X(t))dW_t$ ….(5)

Can be determined using the Picard stochastic frequency, which is a direct extension of the inevitable Picard frequency. Thus, we rewrite the equation in an integrated form

$X(t) = X_0 + \int_0^t b(s, X(s))ds + \int_0^t B(s, X(s))dW(s)$ …..(6)

Then the solution can be approximated with the next iteration.

Ito's integral formula [1],[7]:

Consider the Ito's stochastic differential equation in the form

$dX(t) = b(t, X(t))dt + B(t, X(t))dW(t)$ ….(7)

for $0 \le t \le T$, let $F(t, X(t))$ be a smooth function, : by(main Taylor rule) we have then :

$dF(t, X(t)) = \left(\frac{\partial F}{\partial t} + b(t, X(t))\frac{\partial F}{\partial x} + \frac{1}{2}B^2(t, X(t))\frac{\partial^2 F}{\partial x^2}\right)dt$
$+B(t, X(t))\frac{\partial F}{\partial x}dW(t)$ …(8)

Eq.(8) is called Ito's formula where $X(t)$ satisfies eq.(7) .

Remark 1:

The proposed Intermediate points τ_k which is a generalization to Ito formula and Stratonovich , given by :

Definition[10]: Let be $[0, T] \subset \mathbb{R}$ and $\Delta = \{t_0 = 0 < t_1 < t_2 < \cdots < t_m = T\}$ division of $[0, T]$ with norm $\|\Delta\| = \max_{k\in\{0,1,\dots,m-1\}} (t_{k+1} - t_k)$ and Intermediate points $\tau_k = \sum_{k=0}^{n-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i(t_k + t_{k+1})}{j}$, $i \le j$ then $\int_0^t B(s, X(s))dW(s) = \lim_{\|\Delta\|\to 0}\sum_{k=0}^{m-1} B(\tau_k, X(\tau_k))(W(t_{k+1}) - W(t_k))$…(9)

Special case :

(a) Ito integral definition: taking in (9) $i = 1, j = 1 \Rightarrow \tau_k = t_k + t_{k+1} = t_{k+1}$.

(b) Stratonovich integral definition: taking in (9) $i = 1, j = 2$
$\Rightarrow \tau_k = (t_k + t_{k+1})/2$.

For , $i \le j$ and partition Δ of $[0, T]$ then the stochastic integral is :

$\int_0^T W(s)dW(s) = \lim_{\|\Delta\|\to 0}\sum_{k=0}^{m-1} W(\tau_k)(W(t_{k+1}) - W(t_k))$ …(10)

with $\tau_k = \sum_{k=0}^{n-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i(t_k + t_{k+1})}{j}$, $i \le j$.

If we denote : $R_m = \sum_{k=0}^{m-1} W(\tau_k)(W(t_{k+1}) - W(t_k))$ then

$\lim_{m\to\infty} R_m = \frac{W(t)}{2} + \left(\frac{i}{j} - \frac{1}{2}\right)T$ …(11)

From (10) and (11) one can find the relationships:

(a) for Ito stochastic integral: $\int_0^T W(s)dW(s) = \frac{1}{2}W(t) - \frac{1}{2}T$

(b) for Stratonovich stochastic integral (denoted with " ∘"): $\int_0^T W(s) \circ dW(s) = \frac{1}{2}W(t)$,

that cross the relationship between the two integrals .

More generally , one dimensional Ito-Stratonovich conversion formula is given by:

$\int_0^T B(t, W) \circ dW = \int_0^T B(t, W)dW + \frac{1}{2}\int_0^T \frac{\partial B}{\partial x}(t, W)dt$ ….(12)

The Exact solution for the harmonic SDE :

Ito formula is used to determined exact solution for harmonic stochastic differential equations .

Consider the stochastic differential equation

$$\begin{cases} dX_t = b(t, X_t)dt + B(t, X_t)dW_t \\ X_{t_0} = X_0 \end{cases} \quad …..(13)$$

The proposed model :

For a given $t_0 < t_1 < \cdots = T$, $b(.)$ and B(.) some harmonic stochastic differential equation . such as

$$\left.\begin{aligned} b(t, X_t) &= a\, sin(X_t)\, cos(X_t) + \frac{a^2}{2}\sin(X_t)\cos(X_t)\cos(2X_t) \\ B(t, X_t) &= a\, sin(X_t)\, cos(X_t) \end{aligned}\right\} \quad ….(14)$$

Where $a = 1$, $t \in [0,1]$ and $X_0 = 0.7282$ are arbitrary values. then the true solution by using Ito formula is :

$X_t = tan^{-1}(tanX_0 e^{a(t+W_t)})$ …(15)

Approximate solution :

We use some numerical approximate such as the Euler-Maruyama method and the Milstein method and also calculate the error to explain the convergence of the solution the exact and approximation. The Methods are also distinguished with respect to whether they are 'strongly' or 'weakly' convergent .

Consider the given stochastic differential equation

$$dX_t = \left(a\sin(X_t)\cos(X_t) + \frac{a^2}{2}\sin(X_t)\cos(X_t)\cos(2X_t)\right)dt + a\sin(X_t)\cos(X_t)\,dW_t \quad …..(16)$$

Where $a = 1$ *and* $X_0 = 0.7282$ *are arbitrary values* .

Euler Maruyama method has the form [11] :-

$X_{t_{n+1}} = X_{t_n} + b_n(t_{n+1} - t_n) + B_n(W_{t_{n+1}} - W_{t_n})$ …(17)

$$X_{t_{i+1}} = X_{t_i} + \left[a\sin(X_t)\cos(X_t) + \frac{a^2}{2}\sin(X_t)\cos(X_t)\cos(2X_t)\right]\Delta t + a\sin(X_t)\cos(X_t)\sqrt{\Delta t}\eta_i \quad …(18)$$

With $X_0 = 0.7282$ and $a = 1$, $t_i = i\Delta t \Rightarrow \Delta t = \frac{1}{N}$, $\eta_i \sim (0,1)$

Milstein's method has the form[12] :-

$$X_{t_{n+1}} = X_{t_n} + b_n(t_{n+1} - t_n) + B_n\left(W_{t_{n+1}} - W_{t_n}\right) + \frac{1}{2} b_n b_{X_{t_n}}[(dW_t)^2 - dt] \ldots(19)$$

$$X_{t_{i+1}} = X_{t_i} + \left[a \sin(X_t)\cos(X_t) + \frac{a^2}{2}\sin(X_t)\cos(X_t)\cos(2X_t)\right]\Delta t + a\sin(X_t)\cos(X_t)\sqrt{\Delta t}\eta_i + \frac{a^2}{2}\sin(X_t)\cos(X_t)\cos(2X_t)(\eta_i^2 - 1)\Delta t \ldots(20)$$

The Tables and figures below using Maple Programming Language **[13]** explain the results obtained for the true and approximate solution for the given harmonic stochastic differential equation with N=10 .

Table 1: Analysis of the comparative results of Euler-Maruyama and Millstein's Method.

T	Exact Solution	Euler M. Method	Milstein Method
0.1	0.77824	0.78087	0.77804
0.2	0.82825	0.83126	0.82820
0.3	0.87783	0.87893	0.87817
0.4	0.92649	0.92365	0.92748
0.5	0.97382	0.96535	0.97562
0.6	1.01943	1.00406	1.02221
0.7	1.06203	1.03991	1.06685
0.8	1.10440	1.07308	1.10928
0.9	1.14330	1.10375	1.14928
1.0	1.17987	1.13213	1.18673

Table 2: Error analysis for Euler- Maruyama and Millstein method

T	Exact Solution	Euler M. Method	Milstein Method
0.1	0.77824	0.00263	0.00020
0.2	0.82825	0.00301	0.00005
0.3	0.87783	0.00110	0.00034
0.4	0.92649	0.00284	0.00099
0.5	0.97382	0.00847	0.00180
0.6	1.01943	0.01537	0.00278
0.7	1.06203	0.02212	0.00482
0.8	1.10440	0.03132	0.00488
0.9	1.14330	0.03955	0.00598
1.0	1.17987	0.04774	0.00686

Fig. 1: Shows the graph of the result on Table 1

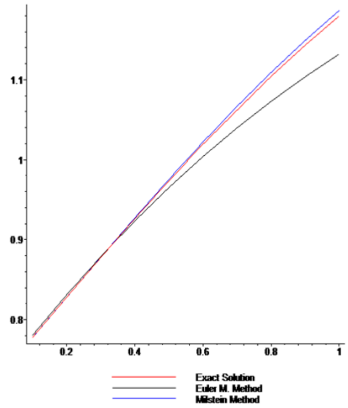
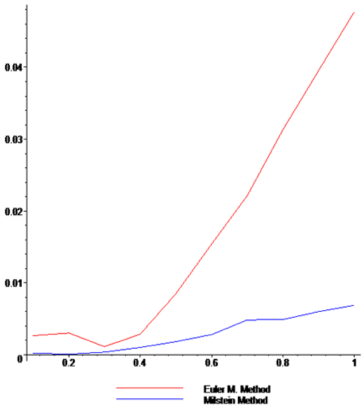


Fig. 2: Shows the graph of the result on Table 2



From the above tables and figures, Milstein method is more accurate than Euler Maruyama method. For Milstein method, the errors decrease as N increases. The error incurred in Euler Maruyama method increases for N =10. The numerical results can confirm that the performance of Milstein method is much better the original one. Overall, Milstein method performed consistently better than Euler Maruyama method with respect to accuracy.

The accuracy of Convergence:[14][15]:

W explain the accuracy of a numerical method for simulating of a given harmonic stochastic differential equaion is measured by how speedy the numerical solution tends toward the exact solution as the time step, Δt, is reduced and a specific a sample path, the error of X_T scales as some power of the time step. The higher this power is, the faster the rate of convergence, the faster the rate of convergence, the larger the time steps that can be used to achieve a specific accuracy, the larger the time step, the fewer steps are required for simulation .

Let us go back to the following SDE $dX_t = \left(a\sin(X_t)\cos(X_t) + \frac{a^2}{2}\sin(X_t)\cos(X_t)\cos(2X_t)\right)dt + a\sin(X_t)\cos(X_t)\,dW_t$

We know that the exact solution is

$$X_t = tan^{-1}(tanX_0 e^{a(t+W_t)})$$

Now after we approximate the solution by the two methodes (through any numerical scheme) and got solution $X^m[N]$.

In order to get some idea about the approximation error; , We use 'M' *as* independent runs of the algorithm and compute

$$\frac{1}{M}\sum_{m=1}^{M}\left|tan^{-1}(tanX_0 e^{a(t+W_t)}) - X^m[N]\right| \approx E|X_t - X[N]|$$

Or

$$\left|E\left(tan^{-1}(tanX_0 e^{a(t+W_t)})\right) - \frac{1}{M}\sum_{m=1}^{M} X^m[N]\right| \approx |E(X_t) - E(X[N])|$$

Then, numerical approximation is strongly convergent with time step δt if :

$$\lim_{\delta t\to 0} E\left|X_t - X_t^{\delta t}\right| = 0 \quad ....(21)$$

And weakly convergent if :

$\lim_{\delta t\to 0}\left|Eg(X_t) - Eg(X_t^{\delta t})\right| = 0,$ for every polynomial 'g' ....(22)

Also, the numerical method is strongly convergent with order γ if :

$$E\left|X_t - X_t^{\delta t}\right| \le K_t(\delta t)^{\gamma} \quad ....(23)$$

K_t is constant depends on 't' and the considered SDE.

And weakly convergent with order γ if :

$$\left|Eg(X_t) - Eg(X_t^{\delta t})\right| \le K_t^{g}(\delta t)^{\gamma} \quad ...(24)$$

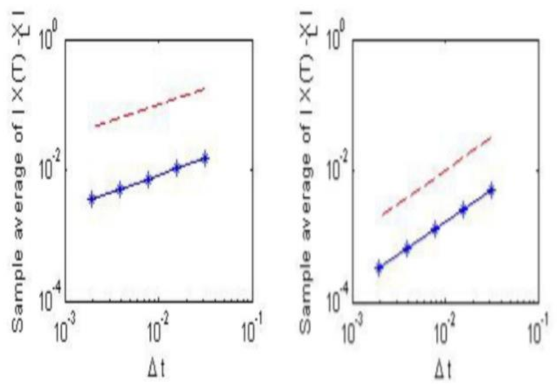
For every polynomial g, the constant K_t^g depends on 't' , g , and the considered SDE.If a numerical scheme is convergent with order γ and we make the steps $'k'$ times smaller then the approximation error will decrease by a factor of k^{γ} .

Remark2:

(a) Euler Maruyama scheme is called strongly convergent if $\gamma = 0.5$ and Euler Maruyama scheme is called weakly convergent if $\gamma = 1$.

(b) Milstein scheme is called strongly convergent if $\gamma = 1$ and Milstein scheme is called weakly convergent if $\gamma = 1$.

Figure 3: Using Matlab Programming Language [5] we find Strong error plots: dashed red line is the appropriate reference slope in each case. left is for Euler-Maruyama and right is for Milstein. Slops of red dashed lines are (0.5109)and(0.9980) respectively .



Conclusion

In this paper, we examined the convergence of solutions between Euler- Maruyama and Milstein numerical methods for stochastic differential equations. One of the simplest numerical methods for solving stochastic differential equations is the Euler-Maruyama method, but one of the downsides of the convergence method is very slow. The Milstein method is a Taylor method, meaning that it is derived from the truncation of the stochastic Taylor expansion of the solution. In some cases this is a disadvantage, as the partial derivative appears in the approximation method and must be provided by the user explicitly. In general, the Euler- Maruyama method is slowly convergence with the stochastic solution process and less accurate than the Milstein method as we can see from the tables and the figure above. For The proposed model, using the Euler-Maruyama method we obtained strong of convergence above the value predicted by theory and the convergence results for the Milstein scheme were in good accordance with the theoretically predicted values .

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تقارب الحل لبعض المعادلات التفاضلية التصادفية التوافقية مع تطبيق

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الملخص

الغرض من هذه البحث هو تقديم مقدمة لنظرية وحساب وتطبيق المعادلات التفاضلية العشوائية كما ندرس المعادلات التفاضلية التصادفية في شكل ملخص وكذلك ندرس الحل المضبوط والحل التقريبي لبعض المعادلات التفاضلية التصادفية التوافقية، باستخدام صيغة أيتو التكاملية والتقريب العددي (طريقة أويلر–ماروياما وطريقة ميلستين) من أجل مناقشة دقة تقارب حلها. وايضاً اقترحنا نقاط وسيطة τ_k للتعميم على صيغة أيتو التكاملية وصيغة ستراتونوفيش. طريقة ميلستين أكثر دقة من طريقة أويلر–ماروياما . من خلال النظر إلى معدلات التقارب في كلتا الطريقتين، نجد أن طريقة أويلر–ماروياما تتقارب بقوة مع $\gamma = 0.5$ وتتقارب بشكل ضعيف مع $\gamma = 1$، في حين أن طريقة ميلستين تتقارب بقوة وضعف مع $\gamma = 1$.