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Some common fixed point theorems of p- contraction and (α, β) composition contraction in Generalized Banach space

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ABSTRACT

In this work, we introduce Some common fixed point theorems of $p-$ contraction and (α,β) composition contraction in Generalized Banach space. The provided theorem is a generalization and extension of many well-known theorems.

1. Introduction

Banach [2] showed and proved a crucial conclusion in complete metric spaces, namely that each contraction mapping of a complete metric space has into itself a unique fixed point. Bakhtin [1] and Czerwik [7], as a generalization of metric spaces, developed the notion of b- metric spaces in 1989. For the study of fixed points in standard metric spaces or other generalized metric spaces, many academics extended and confirmed this theory, we recommend (M. Boriceanu [4], Bota et al. [5], Ding et al. [8], al. [10], Ozturk et al. [12]. A. Harandi, Mehmet Kir et al., Many researchers, on the other hand, have used rational type contractive conditions (Khan et al. [11], Sarwar and Rahman [13], S. Xie et al. [15] and[14] Merdaci, S., and T. Hamaizia). In this work, we show and expand some common fixed point theorems that are also valid in generalized Banach space, we also provide some specific examples to show the veracity of our results.

A mapping $F : H \rightarrow H$ where $(H, \|.\|)$ is a Banach space, is said to be a contraction if there exists $0 \leq k < 1$ such that, for all $x, y \in H$,

$$\|Fx - Fy\| \leq z\|x - y\|. \dots (1.1)$$

Definition 1.1[3] :If H nonempty is a linear space having $s \geq 1$, let $\|.\|$ denotes a function from linear space H into $\mathbb{R}$ that satisfies the following axioms:

1. for all $x \in H$ $\|x\| \geq 0$, $\|x\| = 0$ if and only if $x = 0$;
2. for all $x, y \in H$, $\|x + y\| \leq s[\|x\| + \|y\|]$;
3. for all $x \in H, \alpha \in \mathbb{R}$, $\|\alpha x\| = |\alpha|\,\|x\|$;

$(H, \|.\|)$is called generalized normed linear space. If for $s = 1$, it reduces to standard normed linear space.

Definition 1.2[3] Let $(H, \|.\|)$ be a generalized normed space then the sequence $\{u_n\}$ in H is called,

1. Cauchy sequence iff for each $\varepsilon > 0$, there exist $n(\varepsilon) \in \mathbb{N}$ such that for all $m, n \geq n(\varepsilon)$ we have $\|u_n - u_m\| < \varepsilon$.
2. Convergent sequence iff there exist $u \in H$ such that for all $\varepsilon > 0$, there exist $n(\varepsilon) \in \mathbb{N}$ such that for every $n \geq n(\varepsilon)$ we have $\|u_n - u\| < \varepsilon$.

Definition 1.3[3]: generalized Banach space is a linear generalized normed space in which every Cauchy sequence is convergent.

Lemma 1.1.[6] Let $0 \leq p < 1$. And $u, v \geq 0$ Then the following inequality is holds

$(u+v)^p \leq u^p + v^p$, …(1.2)

Lemma 1.2.[9] Let $(H, \|.\|)$ be a G.B.S. with a real number $s \geq 1$, and F self-mapping on H, assume that $\{u_n\}$ is a sequence in H defined by $u_{n+1} = Fu_n$ if,

$\|u_n - u_{n+1}\| \leq \alpha \|u_{n-1} - u_n\|$, for all $n \in \mathbb{N}$ …(1.3)

where $\alpha \in [0,1)$, $0 \leq s\alpha < 1$. Then $\{u_n\}$ is a Cauchy sequence and is a converges to some $u^* \in$ H as $n \to +\infty$.

2. Main Result

Definition 2.1: Let H be a G.B.S. with $\|.\|$, $s \geq 1$ and let $F, Q : H \to H$ is said to be $p-$ contraction, such that

$$\|Fx - Qy\|^P \leq a \frac{\|Fx-x\|^P + (\|x-y\|\|Qy-y\|)^P}{1+ \|Fx-Qy\|^P} + b\frac{(\|x-Qy\| \|x-Fx\|)^P}{(\|x-y\| + \|Fx-Qy\|)^P} + c\frac{\|x-Qy\|^P}{1+\|y-Fx\|^P} \ldots(2.1)$$

for all $x, y \in H$, $0 < P \leq 1$ where a, b and c are nonnegative real numbers and $(0 \leq a + b\, s^p + 2c\, s^p \leq 1)$, $(1 \leq a\, s^p < 1)$.

Theorem 2.1: Let $(H, \|.\|)$ be a G.B.S. and let $F, Q: H \to H$ be a two mappings a $p-$ contraction. Then F and Q have a unique common fixed point.

Proof: We define a sequence. $\{u_n\}$ in H such that $u_{n+1} = Fu_n$, $u_{n+2} = Qu_{n+1}$, for all $n \in \mathbb{N}$. ./../(2.2)

(I) We'll show that $\{u_n\}$ is Cauchy sequence,

If, $u_n = u_{n+1}$ for all $n \in \mathbb{N}$, and let $n = 2z$, then $u_{2z} = u_{2z+1}$ as well as the condition(2.1) with $x = u_{2z}$ and $y = u_{2z+1}$ we have,

$$\|u_{2z+1} - u_{2z+2}\|^P = \|Fu_{2z} - Qu_{2z+1}\|^P$$
$$\leq a \frac{\|Fu_{2z}-u_{2z}\|^P + (\|u_{2z}-u_{2z+1}\|\|Qu_{2z+1}-u_{2z+1}\|)^P}{1+\|Fu_{2z}-Qu_{2z+1}\|^P} + b\frac{(\|u_{2z}-Qu_{2z+1}\| \|u_{2z}-Fu_{2z}\|)^P}{(\|u_{2z}-u_{2z+1}\| + \|Fu_{2z}-Qu_{2z+1}\|)^P} + c\frac{\|u_{2z}-Qu_{2z+1}\|^P}{1+\|u_{2z+1}-Fu_{2z}\|^P}$$
$$= a\frac{\|u_{2z+1}-u_{2z}\|^P + (\|u_{2z}-u_{2z+1}\|\|u_{2z+2}-u_{2z+1}\|)^P}{1+\|u_{2z+2}-u_{2z+1}\|^P} + b\frac{(\|u_{2z}-u_{2z+2}\| \|u_{2z}-u_{2z+1}\|)^P}{(\|u_{2z}-u_{2z+1}\| + \|u_{2z+1}-u_{2z+2}\|)^P} + c\frac{\|u_{2z}-u_{2z+2}\|^P}{1+\|u_{2z+1}-u_{2z+1}\|^P}$$
$$= c\|u_{2z} - u_{2z+2}\|^P$$
$$\leq c\,(s(\|u_{2z} - u_{2z+1}\| + \|u_{2z+1} - u_{2z+2}\|))^P,$$

by Property (2) from definition (1.1) and by condition (2.1) and by lemma [1.1], we get,

$$\|u_{2z+1} - u_{2z+2}\|^P \leq c(s)^p\, [\,(\|u_{2z} - u_{2z+1}\| + \|u_{2z+1} - u_{2z+2}\|)^p]$$
$$\leq c\,(s)^p\, [\, \|u_{2z} - u_{2z+1}\|^p + \|u_{2z+1} - u_{2z+2}\|^p]$$
$$\|u_{2z+1} - u_{2z+2}\|^p \leq c\,(s)^p\, \|u_{2z+1} - u_{2z+2}\|^p,$$

raising two sides of the inequality to the power of $1/p$,

which is a contradiction. Because $0 \leq c\, s = \gamma < 1$,

$$\|u_{2z+1} - u_{2z+2}\| \leq \gamma_1 \|u_{2z+1} - u_{2z+2}\|, \ldots (2.3)$$

Now, if $n = 2z + 1$ then using the identical arguments as were used in the case $u_{2z} = u_{2z+1}$, it can be shown that from now on, we assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$.

$$\|u_{2z+1} - u_{2z+2}\|^P = \|Fu_{2z} - Qu_{2z+1}\|^P$$
$$\leq a \frac{\|Fu_{2z}-u_{2z}\|^P + (\|u_{2z}-u_{2z+1}\|\|Qu_{2z+1}-u_{2z+1}\|)^P}{1+\|Fu_{2z}-Qu_{2z+1}\|^P} + b\frac{(\|u_{2z}-Qu_{2z+1}\| \|u_{2z}-Fu_{2z}\|)^P}{(\|u_{2z}-u_{2z+1}\| + \|Fu_{2z}-Qu_{2z+1}\|)^P} + c\frac{\|u_{2z}-Qu_{2z+1}\|^P}{1+\|u_{2z+1}-Fu_{2z}\|^P}$$
$$= a\frac{\|u_{2z+1}-u_{2z}\|^P + (\|u_{2z}-u_{2z+1}\|\|u_{2z+2}-u_{2z+1}\|)^P}{1+\|u_{2z+2}-u_{2z+1}\|^P} + b\frac{(\|u_{2z}-u_{2z+2}\| \|u_{2z}-u_{2z+1}\|)^P}{(\|u_{2z}-u_{2z+1}\| + \|u_{2z+1}-u_{2z+2}\|)^P} + c\frac{\|u_{2z}-u_{2z+2}\|^P}{1+\|u_{2z+1}-u_{2z+1}\|^P}$$
$$= a\frac{\|u_{2z+1}-u_{2z}\|^P + (\|u_{2z}-u_{2z+1}\|\|u_{2z+2}-u_{2z+1}\|)^P}{1+\|u_{2z+2}-u_{2z+1}\|^P} + b\frac{(\|u_{2z}-u_{2z+2}\| \|u_{2z}-u_{2z+1}\|)^P}{(\|u_{2z}-u_{2z+1}\| + \|u_{2z+1}-u_{2z+2}\|)^P} + c\frac{\|u_{2z}-u_{2z+2}\|^P}{1+\|u_{2z+1}-u_{2z+1}\|^P}$$
$$= a\frac{\|u_{2z+1}-u_{2z}\|^P\,(1+\|u_{2z}-u_{2z+1}\|^P)}{1+(\|u_{2z}-u_{2z+1}\|)^P} + b\frac{(\|u_{2z}-u_{2z+2}\| \|u_{2z}-u_{2z+1}\|)^P}{(\|u_{2z}-u_{2z+1}\| + \|u_{2z+1}-u_{2z+2}\|)^P} + c\|u_{2z} - u_{2z+2}\|^P$$

by condition (2.1) from lemma 1.1, we get

$$\leq a \|u_{2z+1} - u_{2z}\|^P + b\frac{s^p\, \|u_{2z}-u_{2z+2}\|^p \|u_{2z}-u_{2z+1}\|^p}{\|u_{2z}-u_{2z+2}\|^p} + c\,(s[\|u_{2z} - u_{2z+1}\| + \|u_{2z+1} - u_{2z+2}\|])^p$$

Since, $\left(\frac{1}{[\,(\|u_{2z}-u_{2z+1}\| + \|u_{2z+1}-u_{2z+2}\|)^p\,]} \leq \frac{s^p}{\|u_{2z}-u_{2z+2}\|^p}\right.$ such that $[\|u_{2z} - u_{2z+2}\| \neq 0$ and $(\|u_{2z} - u_{2z+1}\| + \|u_{2z+1} - u_{2z+2}\|)^p \neq 0\,]$,

we get,

$$\|u_{2z+1} - u_{2z+2}\|^p \leq a \|u_{2z+1} - u_{2z}\|^p + b\, s^p \|u_{2z} - u_{2z+2}\|^p + c\, s^p\, [\,\|u_{2z} - u_{2z+1}\|^p + \|u_{2z+1} - u_{2z+2}\|^p\,]$$
$$\leq a \|u_{2z+1} - u_{2z}\|^p + b\, s^p \|u_{2z} - u_{2z+2}\|^p + c\, s^p\, [\,\|u_{2z} - u_{2z+1}\|^p + c\, s^p \|u_{2z+1} - u_{2z+2}\|^p\,]$$
$$(1 - c\, s^p)\|u_{2z+1} - u_{2z+2}\|^p \leq (a + b\, s^p + c\, s^p)\, \|u_{2z} - u_{2z+1}\|^p$$
$$\|u_{2z+1} - u_{2z+2}\|^p \leq \frac{(a + b\, s^p + c\, s^p)}{(1 - c\, s^p)} \|u_{2z} - u_{2z+1}\|^p.$$

Raising two sides of the inequality to the power of 1/p.

We get,

$$\|u_{2z+1} - u_{2z+2}\| \leq \left(\frac{(a + b\, s^p + c\, s^p)}{(1 - c\, s^p)}\right)^{\frac{1}{p}} \|u_{2z} - u_{2z+1}\|$$
$$\|u_{2z+1} - u_{2z+2}\| \leq \gamma_2 \|u_{2z} - u_{2z+1}\|, \ldots(2.4)$$
$$\gamma = \left(\frac{(a + b\, s^p + c\, s^p)}{(1 - c\, s^p)}\right)^{\frac{1}{p}} < 1.$$

where $(0 \leq a + b\, s^p + 2c\, s^p \leq 1)$.

From (2.3)and (2.4) by using lemma (1.2) limit $n \to \infty$ we can deduce that $\{u_n\}$ is a Cauchy sequence in $(H, \|.\|)$, and that is a converges to some $u^* \in$ H as $n \to +\infty$.

(II) We will prove that $Fu^* = Qu^* = u^*$.

by property (2) from definition (1.1) and condition(2.1), we have,

$$\|u^* - Fu^*\|^p \leq (s[\|u^* - u_{2n+2}\| + \|u_{2n+2} - Fu^*\|])^p$$

$\leq s^p\|u^*-u_{2n+2}\|^p + s^p \|Fu^*-Qu_{2n+1}\|^p$

$\leq s^p\|u^*-u_{2n+2}\|^p + s^p a \frac{\|Fu^*-u^*\|^p+(\|u^*-u_{2n+1}\|\|Qu_{2n+1}-u_{2n+1}\|)^p}{1+\|Fu^*-Qu_{2n+1}\|^p} + s^p b\frac{(\|u^*-Qu_{2n+1}\|\,\|u^*-Fu^*\|)^p}{(\|u^*-u_{2n+1}\|+\|Fu^*-Qu_{2n+1}\|)^p} + s^p c \frac{\|u^*-Qu_{2n+1}\|^p}{1+\|u_{2n+1}-Fu^*\|^p}$

$= s^p\|u^*-u_{2n+2}\|^p + s^p a \frac{\|Fu^*-u^*\|^p+(\|u^*-u_{2n+1}\|\|u_{2n+2}-u_{2n+1}\|)^p}{1+\|Fu^*-u_{2n+2}\|^p} + s^p b\frac{(\|u^*-u_{2n+2}\|\,\|u^*-Fu^*\|)^p}{(\|u^*-u_{2n+1}\|+\|Fu^*-u_{2n+2}\|)^p} + s^p c \frac{\|u^*-u_{2n+2}\|^p}{1+\|u_{2n+1}-Fu^*\|^p}$

by take the limit as $n \to +\infty$, we obtain that

$\|u^*-Fu^*\|^p \leq s^p a \frac{\|Fu^*-u^*\|^p}{1+\|Fu^*-u^*\|^p} \leq s^p a \|Fu^*-u^*\|^p$

Since $0 \leq s^p a < 1$, hence $\|u^*-Fu^*\| = 0$, thus $Fu^* = u^*$.

Similarly, we obtain

$\|u^*-Qu^*\|^p \leq (s[\|u^*-u_{2n+1}\| + \|u_{2n+1}-Qu^*\|])^p$

$\leq s^p\|u^*-u_{2n+1}\|^p + s^p \|Fu_{2n}-Qu^*\|^p$

$\leq s^p\|u^*-u_{2n+2}\|^p + s^p a \frac{\|Fu_{2n}-u_{2n}\|^p+(\|u_{2n}-u^*\|\|Qu^*-u^*\|)^p}{1+\|Fu_{2n}-Qu^*\|^p} + s^p b\frac{(\|u_{2n}-Qu^*\|\,\|u_{2n}-Fu_{2n}\|)^p}{(\|u_{2n}-u^*\|+\|Fu_{2n}-Qu^*\|)^p} + s^p c \frac{\|u_{2n}-Qu^*\|^p}{1+\|u^*-Fu_{2n}\|^p}$

$= s^p\|u^*-u_{2n+2}\|^p + s^p a \frac{\|u_{2n+1}-u_{2n}\|^p+(\|u_{2n}-u^*\|\|Qu^*-u^*\|)^p}{1+\|u_{2n+1}-Qu^*\|^p} + s^p b\frac{(\|u_{2n}-Qu^*\|\,\|u_{2n}-u_{2n+1}\|)^p}{(\|u_{2n}-u^*\|+\|u_{2n+1}-Qu^*\|)^p} + s^p c \frac{\|u_{2n}-Qu^*\|^p}{1+\|u^*-u_{2n+1}\|^p}$

by take the limit as $n \to +\infty$, we obtainthat,

$\|u^*-Qu^*\|^p \leq s^p c \|u^*-Qu^*\|^p$ which is contradiction. $(0 \leq s^p c < 1)$

$\|u^*-Qu^*\| = 0$, thus $Qu^* = u^*$.

Thus u^* is a common fixed point of F and Q.

(III) We'll show that F and Q have a unique common fixed point.

Suppose now that u^* and v^*, are different common fixed points of F and Q, then from (2.1), we have

$\|u^*-v^*\|^p = \|Fu^*-Qv^*\|^p$

$\leq a \frac{\|Fu^*-u^*\|^p+(\|u^*-v^*\|\|Qv^*-v^*\|)^p}{1+\|Fu^*-Qv^*\|^p} + b\frac{(\|u^*-Qv^*\|\,\|u^*-Fu^*\|)^p}{(\|u^*-v^*\|+\|Fu^*-Qv^*\|)^p} + c\frac{\|u^*-Qv^*\|^p}{1+\|v^*-Fu^*\|^p}$

$= a \frac{\|u^*-u^*\|^p+(\|u^*-v^*\|\|v^*-v^*\|)^p}{1+\|u^*-y^*\|^p} + b\frac{(\|u^*-v^*\|\,\|u^*-u^*\|)^p}{(\|u^*-v^*\|+\|u^*-v^*\|)^p} + c\frac{\|u^*-v^*\|^p}{1+\|v^*-u^*\|^p}$

$= c\frac{\|u^*-v^*\|^p}{1+\|u^*-v^*\|^p}$

$\|u^*-v^*\|^p \leq c\frac{\|u^*-v^*\|^p}{1+\|u^*-v^*\|^p} \leq c\|u^*-v^*\|^p$, $(0 \leq c < 1)$,

we have $\|u^*-v^*\| = 0$, i.e. $u^* = v^*$.

We proved that F and Q have a unique common fixed point.

Corollary 2.1. Let H be a G.B.S. with $\|.\|$ and let $F,Q: H \to H$ be a two mappings on H fulfilling the condition, $s \geq 1$ and $(0 \leq p < 1)$

$\sqrt{\|Fx-Qy\|} \leq a \frac{\sqrt{\|Fx-x\|}+\sqrt{\|x-y\|\|Qy-y\|}}{1+\sqrt{\|Fx-Qy\|}} + b\frac{\sqrt{\|x-Qy\|\,\|x-Fx\|}}{\sqrt{\|x-y\|+\|Fx-Qy\|}} + c\frac{\sqrt{\|x-Qy\|}}{1+\sqrt{\|y-Fx\|}}$, … (2.5)

for all $x, y \in H$, where a, b and c are nonnegative .real numbers. and

$(0 \leq a + b\sqrt{s} + 2c\sqrt{s} < 1)$. Then F and Q have a unique common fixed point.

Corollary 2.2. Let H be a G.B.S. with $\|.\|$ and let $F, Q : H \to H$ be a two mappings on H fulfilling the condition, $s \geq 1$ and $(0 \leq p < 1)$

$\|Fx-Qy\|^P \leq a \frac{\|Fx-x\|^P+(\|x-y\|\|Qy-y\|)^P}{1+\|Fx-Qy\|^P} + b\frac{(\|x-Qy\|\,\|x-Fx\|)^P}{(\|x-y\|+\|Fx-Qy\|)^P}$, … (2.6)

for all $x, y \in H$, where a, b and c are nonnegative .real numbers. and

$(0 \leq a + b\sqrt[p]{s} < 1)$. Then F and Q have a unique common fixed point.

Corollary 2.3.Let H be a G.B.S. with $\|.\|$ and let $F,Q: H \to H$ be a two mappings on H fulfilling the condition, $s \geq 1$ and $(0 \leq p < 1)$

$\|Fx-Qy\|^P \leq a \frac{\|Fx-x\|^P+(\|x-y\|\|Qy-y\|)^P}{1+\|Fx-Qy\|^P} + b\frac{\|x-Qy\|^P}{1+\|y-Fx\|^P}$, ..(2.7)

for all $x, y \in H$, where a, b and c are nonnegative .real numbers. and

$(0 \leq a + 2b\sqrt[p]{s} < 1)$. Then F and S have a unique common fixed point.

Definition 2.2: Let H be a G.B.S. with $\|.\|$ and $s \geq 1$ a mappings $F,Q: H \to H$ is said be a (α,β) composition contraction if fulfilling the condition,

$\|QFx-FQy\| \leq \alpha\|Fx-FQy\| + \beta\|Qy-QFx\|$, …(2.8)

for all $x, y \in H$, where α, β and c are nonnegative real numbers, $\alpha s < \frac{1}{2}$ and $s\beta < 1$.

Theorem 2.2: Let H be a G.B.S. with $\|.\|$ and $F,Q: H \to H$ a (α,β) composition contraction. Then QFx and FQy have a unique common fixed point.

Proof: We define a sequence.$\{u_n\}$ in H as a result,

$Fu_n = u_{n+1}$, $Qu_{n+1} = u_{2n+2}$, $QFu_n = u_{n+2}$ and $FQu_{n+1} = u_{n+3}$, (2.9)

for all $n \in \mathbb{N}$.

If, $n = 2z$ for all $n \in \mathbb{N}$, as well as the (3.20) with $x = u_{2z}$ and $y = u_{2z+1}$ we have,

$\|u_{2z+2}-u_{2z+3}\| = \|QFu_{2z}-FQu_{2z+1}\|$

$\leq \alpha\|Fu_{2z}-FQu_{2z+1}\| + \beta\|Qu_{2z+1}-QFu_{2z}\|$

$= \alpha\|u_{2z+1}-u_{2z+3}\| + \beta\|u_{2z+2}-u_{2z+2}\| = \alpha\|u_{2z+1}-u_{2z+3}\|$

by property(2) of the definition(2.1), and the condition (2.9), we have

$\|u_{2z+2} - u_{2z+3}\| \leq \alpha s[\ \|u_{2z+1} - u_{2z+2}\| + \|u_{2z+2} - u_{2z+3}\|\ ]$

$\|u_{2z+2} - u_{2z+3}\| \leq \frac{\alpha s}{1-\alpha s}\ \|u_{2z+1} - u_{2z+2}\|$

$\|u_{2Z+2} - u_{2Z+3}\| \leq \gamma\|u_{2z+1} - u_{2z+2}\|$, let $\gamma = \frac{\alpha s}{1-\alpha s} < 1$.

Using lemma (1.2) limit $n \to \infty$ we can deduce that $\{u_n\}$ is a Cauchy sequence in $(H, \|.\|)$, and that is a converges to some $u^* \in$ H as $n \to +\infty$.

(II) We will prove that $Fu^* = Qu^* = u^*$.

by property(2) of the definition(2.1), and the condition (2.9), we have

$\|u^* - FQu^*\| \leq s[\|u^* - u_{2z+2}\| + \|u_{2z+2} - FQu^*\|]$

$= s\ \|u^* - u_{2z+2}\| + s\ \|QFu_{2z} - FQu^*\|$

$\leq s\ \|u^* - u_{2z+2}\| + s\ \alpha\|Fu_{2z} - FQu^*\| + \beta\|Qu^* - QFu_{2z}\|$

$= s\ \|u^* - u_{2z+2}\| + s\ \alpha\|u_{2z+1} - FQu^*\| + \beta\|Qu^* - Qu_{2z+1}\|$,

by take the limit as $z \to +\infty$, we get,

$\|u^* - FQu^*\| \leq s\alpha\ \|u^* - FQu^*\|$, which is a contradiction because $0 \leq s\alpha < 1$.

Thus $\|u^* - FQu^*\| = 0$, $FQu^* = u^*$,

Similarly, we get,

$\|u^* - QFu^*\| \leq s[\|u^* - u_{2n+3}\| + \|u_{2n+3} - QFu^*\|]$

$= s[\|u^* - u_{2n+3}\| + \|FQu_{2n+1} - QFu^*\|]$

$= s\|u^* - u_{2n+3}\| + s[\ \alpha\|Fu^* - FQu_{2n+1}\| + \beta\|Qu_{2n+1} - QFu^*\|]$,

by take the limit as $n \to +\infty$, we get,

$\|u^* - FQu^*\| \leq s\,\beta\ \|u^* - FQu^*\|$, which is a contradiction,

Because $(0 \leq s\,\beta < 1)$.

Thus $\|u^* - QFu^*\| = 0$, $QFu^* = u^*$.

(III) Now We will show that QFx and FQx have a unique common fixed point.

Assume that $u^*, v^* \in$ H such that, $u^* \neq v^*$ are common fixed points of QFx and FQx, by (3.20), we have

$\|u^* - v^*\| = \|QFu^* - FQv^*\|$

$\leq \alpha\|Fu^* - FQv^*\| + \beta\|Qv^* - QFu^*\|$

$\|u^* - v^*\| \leq (\alpha + \beta)\ \|u^* - v^*\|$, which is contradiction,

Since $0 \leq \alpha + \beta < 1$, we get, $\|u^* - v^*\| = 0$, thus $u^* = v^*$.

Corollary 2.4: Let H be a G.B.S. with $\|.\|$ and let $F, Q: H \to H$ be a two mappings on H fulfilling the condition, and $s \geq 1$,

$\|Fx - Qy\| \leq a\ \|Fx - x\| + sb\frac{\|x-Qy\|}{1+\|Fx-y\|}$, … (2.11)

for all $x, y \in H$ where a, b and c are nonnegative .real numbers. and

$(0 \leq 2s^2b,\ a + 2s^2b,\ as,\ s^2b < 1)$, then F and Q have a unique common fixed point.

Corollary 2.5: Let H be a G.B.S. with $\|.\|$ and let $F, Q: H \to H$ be a two mappings on H fulfilling the condition, and $s \geq 1$,

$\|Fx - Qy\| \leq a\ \|Fx - x\| + c\frac{\|x-Qy\| + \|x-Qy\|\ \|x-Fx\| + \|y-Fx\|\ \|y-Qy\|}{1+\|x-y\|}$, (2.12)

for all $x, y \in H$ where a, b and c are nonnegative .real numbers. and

$(0 \leq 2s^2b + 2sc,\ a + 2sc,\ as < 1)$, then F and Q have a unique common fixed point.

Conclusion:

In generalized Banach space, we introduced some common fixed point theorems, including rational condition of p- contraction and (α, β) composition contraction. The common fixed point contraction mapping principle also has a local version (Corollary (2.1), (2.2), and (2.3) as well as an asymptotic version Theorem 2.1). Also (Corollary (2.4) and (2.5) as well as an asymptotic version Theorem 2.2).

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بعض نظريات النقطة الصامدة المشتركة الانكماشية p و (α, β) التركيب الانكماشي في فضاء باناخ المعمم

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الملخص

في هذا العمل ، نقدم بعض نظريات النقطة الثابتة المشتركة الانكماشية للأس p و ونظرية (α، β) التركيب الانكماشي في فضاء باناخ المعمم. تعتبر النظريات المقدمة هي تعميم وامتداد للعديد من النظريات المعروفة في فضاء باناخ .