IASJ
المجلات الاكاديمية العراقية

Tikrit Journal of Pure Science

ISSN: 1813 – 1662 (Print) --- E-ISSN: 2415 – 1726 (Online)

Journal Homepage: http://tjps.tu.edu.iq/index.php/j

Dynamical Approach in studying GJR-GARCH (Q,P) Models with Application

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https://doi.org/10.25130/tjps.v26i2.131

ARTICLE INFO.

Article history:

-Received: 20 / 12 / 2020

-Accepted: 9 / 2 / 2021

-Available online: / / 2021

Keywords: ARCH model, GARCH model, GJR-GARCH model, Local Linearization method, stability Conditional variance, Heteroscedastic, Limit cycle.

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ABSTRACT

This paper deals with finding stationarity Condition of GJR-GARCH(Q,P) model by using a local linearization technique in order to reduce this non-linear model to a linear difference equation with constant coefficients and then obtain the stationarity condition via a characteristic equation.

Finally we apply the obtained stationarity conditions of GJR-GARCH(Q,P) model to a real data that represents a monthly Brent Crude oil prices at closing in dollars for period (JUN. 1989-DES. 2018) and we find that GJR-GARCH(3,1) is the best model according to AIC and BIC information criteria.

1. Introduction

The stochastic process is 2nd-order stationary if the mean and variance of the process does not depend on time t, the most important condition to the random error in all time series model must be a white noise process with zero mean and constant variance and uncorrelated. In applications the mean and variance may depend on the time t, many researchers discussed this situation and proposed some non-linear time series model known as autoregressive conditional heteroscedastic to avoid the volatility in data that cause the dependence of mean and variance on time t. The first ARCH model proposed by R. Engle in 1982 based on the martingale difference series [2].

A high number of researchers studied the stationarity and existence of a moments for the family of ARCH and GARCH models. Nelson in 1991 proposed an exponential GARCH model (EGARCH) as an alternative symmetric model of logarithmic condition –al variance to avoid the positivity of parameters.[10] Glosten – Jagannathan - Rankle Generalized Autoregressive Conditional Heteroscedasticity Variance model in 1993 which was known for short (GJR-GARCH model) proposed the GJR-GARCH model as an expansion of GARCH model to capture asymmetric impact of negative or positive shocks on the conditional variance usually called the Leverage effect .[4]

Our goal in this paper is to studying the stationarity condition of GJR-GARCH(Q,P) model by using dynamical approach that approximate this model to a linear difference equation, this method known as local linearization approximation method proposed by T. Ozaki (1985) when he find the stability condition of the exponential autoregressive model (EXPAR).[11]. Mohammad and Salim in 2007 used this model in order to find the stability condition of logistic autoregressive model [5], Mohammad and Ghannam in 2010 studying the stability condition of Cauchy model [6], Mohammad and Ghaffar in 2016 studying the stationarity of GARCH(Q,P) model [7], and Mohammad and Mudhir in 2018 studying the EGARCH(Q,P) model [9].

2. preliminaries

A non-linear time series model in a discrete time can be represent as a discrete time dynamical system by considering the system

$$x_t = f(x_{t-1}, x_{t-2}, \dots, x_{t-Q}, z_t) \quad \dots(2.1)$$

where f is a non-linear function and z_t be a random error of the system and its often be a white noise process $(z_t \sim iid\ N(0,\sigma_z^2))$

The ARCH model that proposed by Robert Engle in 1982 with the formula :

$x_t = \sigma_t z_t$ where $z_t \sim iid\ N(0,1)$

$$\sigma_t^2 = w + \sum_{i=1}^{Q} \alpha_i\, x_{t-i}^2 \qquad \ldots(2.2)$$

where $w, \sum_{i=1}^{Q} \alpha_i$ are model parameters, σ_t^2 is the conditional variance.

This model based on martingale difference that is

$E(X_{t+1}^2/F_t) = \sigma_t^2 \quad \ldots(2.3)$

where F_t is a $\sigma - field$ of a random variables $(x_{t-1}, x_{t-2}, \ldots, x_{t-Q})$, sometime called a filter.[3] then (2.2) can be written as a dynamical system :

$\sigma_t^2 = f(w, \sigma_{t-1}^2, \sigma_{t-2}^2, \ldots, \sigma_{t-Q}^2) \quad \ldots(2.4)$

The local linearization technique deals with approximate a non-linear dynamical system to a linear dynamical system in order to study the stability condition of the non-zero fixed point of the original dynamical system. The non-zero fixed point of a function f is also a singular point ξ. If there is no other fixed point in their neighborhood. Sufficient and necessary condition for ξ is satisfy :

$\xi = f(\xi) \quad \ldots(2.5)$

This technique consist of making small perturbation around the non-zero singular point ξ in its neighborhood with a sufficiently small radius ξ_t such that $\xi_t^n \to 0$ for $n \geq 2$. The effect of this small perturbation done by replacing $\xi + \xi_{t-i}$ instead of σ_{t-i}^2 for $1 \leq i \leq Q$ that is mean we use a variational equation :

$\sigma_{t-i}^2 = \xi + \xi_{t-i} \quad$ for $\quad 1 \leq i \leq Q. \quad \ldots(2.6)$

after substituting this variational equation in (2.4) in for example we obtain a linear difference equation of order Q in terms of $\xi_t, \xi_{t-1}, \xi_{t-2}, \ldots, \xi_{t-Q}$ and we can discuss the stability of this linear difference equation via the roots of its characteristic equation.

Lemma 2.1 [7],[9]

Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be a non-negative real numbers, the following polynomial :

$$P(z) = 1 - \sum_{i=1}^{r} \alpha_i\, z^i$$

does not have a roots inside and on the unit circle if and only if $P(z) > 1$.

In 1986 Bollerslev [12] extended the ARCH model and suggested a generalized autoregressive conditional heteroscedasticity model GARCH(Q,P) which has the following formula :

$x_t = \sigma_t z_t$ where $z_t \sim iid\ N(0,1)$

$$\sigma_t^2 = w + \sum_{i=1}^{Q} \alpha_i\, x_{t-i}^2 + \sum_{j=1}^{P} \beta_j\, \sigma_{t-j}^2 \quad \ldots(2.7)$$

Many models where suggested as an expansion of GARCH model. For example a threshold ARCH model, TARCH by Zakoian et al in 1994 [3] and exponential GARCH by Nelson in 1991 [10].

A symmetric power GARCH model has the general form :

$x_t = \sigma_t z_t$ where $z_t \sim iid\ N(0,1)$

$$\sigma_t^\delta = w + \sum_{i=1}^{Q} \alpha_i\,(|x_{t-i}| - \gamma_i x_{t-i})^\delta + \sum_{j=1}^{P} \beta_j\, \sigma_{t-j}^\delta \quad \ldots(2.8)$$

where > 0 ,$\delta > 0, \alpha_i > 0,\ \beta_j > 0$ and $|\gamma_i| \leq 1$ for $1 \leq i \leq Q, 1 \leq j \leq P$.

The GJR-GARCH is a special case of a symmetric power GARCH model where $\delta = 2$ and this model has the form :

$x_t = \sigma_t z_t$ where $z_t \sim iid\ N(0,1)$

$$\sigma_t^2 = w + \sum_{i=1}^{Q} (\alpha_i + \gamma_i I_{x_{t-i}}) x_{t-i}^2 + \sum_{j=1}^{P} \beta_j\, \sigma_{t-j}^2 \quad \ldots(2.9)$$

where $I_{x_{t-i}}$ is indicator function defined as

$$I_{x_{t-i}} = \begin{cases} 0\ if\ x_{t-i} < 0 \\ 1\ if\ x_{t-i} \geq 0 \end{cases} \quad \forall i \qquad \ldots(2.10)$$

[12],[13]

The GJR-GARCH process is stationary if and only if

$$[\sum_{i=1}^{Q}\left(\alpha_i + \frac{\gamma_i}{2}\right) + \sum_{j=1}^{P} \beta_j] < 1 \quad \ldots(2.11)$$

By assumption that the stochastic process of squares $\{x_t^2\}$ is stationary we mean that the variance is constant and independent of t, in this case the variance $\sigma_x^2 = E(x_t^2/F_t) = E(x_{t-i}^2/F_t)$ for $i = 1,2, \ldots, Q$ then be taking the conditional expectation with respect to the filter F_t to both sides of (2.9) we get

$\mathrm{E}(\sigma_t^2/F_t) =$
$E(w/F_t) + \sum_{i=1}^{Q}(\alpha_i + \gamma_i \mathrm{E}(I_{x_{t-i}}))\, \mathrm{E}(x_{t-i}^2/F_t) +$
$\sum_{j=1}^{P} \beta_j\ \mathrm{E}(\sigma_{t-j}^2/F_t)$

$$\sigma_x^2 = w + \sum_{i=1}^{Q}(\alpha_i + \frac{\gamma_i}{2})\, \sigma_x^2 + \sum_{j=1}^{P} \beta_j\, \sigma_x^2 \quad \ldots(2.12)$$

where

$\mathrm{E}(I_{x_{t-i}}) = \int_{-\infty}^{\infty} I_{x_{t-i}} f(x_{t-i}) dx_{t-i}$

$= \int_{-\infty}^{0} 0\, f(x_{t-i}) dx_{t-i} + \int_{0}^{\infty} 1\, f(x_{t-i}) dx_{t-i} = 0 + \frac{1}{2} = \frac{1}{2}.$

Therefore the unconditional variance of the model (2.10) given by

$$\sigma_x^2 = \frac{w}{1 - \left[\sum_{i=1}^{Q}(\alpha_i + \frac{\gamma_i}{2})\ + \sum_{j=1}^{P} \beta_j\ \right]} \quad \ldots(2.13)$$

Of the unconditional variance σ_x^2 exists if

$\left[\sum_{i=1}^{Q}(\alpha_i + \frac{\gamma_i}{2})\ + \sum_{j=1}^{P} \beta_j\ \right] < 1$

Then the stationarity of GJR-GARCH model required that the conditional variance σ_t^2 converges to the unconditional variance σ_x^2 see [13],[9].

The condition (2.11) can be obtained by using a local linearization method as follows since $x_t = \sigma_t z_t$ and $z_t \sim iid\ N(0,1)$, then

$E(x_t) = E(\sigma_t\, z_t) = E(\sigma_t).E(z_t) = E(\sigma_t).0 = 0$

$Var(x_t) = \mathrm{E}(x_t^2) = \mathrm{E}(\sigma_t^2.\ z_t^2) = \sigma_t^2.\mathrm{E}(z_t^2) =$
$\sigma_t^2.1 = \sigma_t^2$

By taking a conditional expectation with respect to the filtering F_{t-i} for $i = 1,2,\dots,Q$ to both sides of the model (2.9) we obtain that

$E(\sigma_t^2/F_t) =$

$E(w/F_t) + \sum_{i=1}^{Q}(\alpha_i + \gamma_i E(I_{x_{t-i}}))\, E(x_{t-i}^2/F_{t-i}) + \sum_{j=1}^{P}\beta_j\, E(\sigma_{t-j}^2/F_{t-j})$

$\sigma_t^2 = w + \sum_{i=1}^{Q}\alpha_i\,\sigma_{t-i}^2 + \sum_{i=1}^{Q}\gamma_i \cdot \frac{1}{2}\cdot\sigma_{t-i}^2 + \sum_{j=1}^{P}\beta_j\,\sigma_{t-j}^2$

$$\therefore\quad \sigma_t^2 = w + \sum_{i=1}^{Q}\alpha_i\,\sigma_{t-i}^2 + \sum_{i=1}^{Q}\gamma_i\cdot\frac{1}{2}\cdot\sigma_{t-i}^2 + \sum_{j=1}^{P}\beta_j\,\sigma_{t-j}^2 \qquad \dots(2.14)$$

In order to find a fixed point we put $\sigma_t^2 = \sigma_{t-1}^2 = \sigma_{t-2}^2 = \dots = \sigma_{t-Q}^2 = \sigma_{t-P}^2 = \xi$

$$\xi = w + \left[\sum_{i=1}^{Q}\left(\alpha_i + \frac{\gamma_i}{2}\right) + \sum_{j=1}^{P}\beta_j\right]\xi \qquad \dots(2.15)$$

$$\therefore\quad \xi = \frac{w}{\left[1 - \sum_{i=1}^{Q}\left(\alpha_i + \frac{\gamma_i}{2}\right) - \sum_{j=1}^{P}\beta_j\right]}$$

Then the non-zero singular point satisfy the condition (2.5) of the GJR-GARCH model is the unconditional variance σ_x^2

Without loos of generality let $r = \max(Q,P)$ then for $Q > P$ we consider $\left(\alpha_r + \frac{\gamma_r}{2}\right) = 0$ for $r = P, P+1, \dots, Q-1$ if for $Q < P$ consider $\beta_r = 0$ for $r = Q, Q+1, \dots P-1$ then the GJR-GARCH model can be written as

$$\sigma_t^2 = w + \sum_{i=1}^{r}(\alpha_i + \gamma_i I_{x_{t-i}})\, x_{t-i}^2 + \sum_{j=1}^{r}\beta_j\,\sigma_{t-j}^2$$

Of the unconditional variance

$$\xi = \sigma_x^2 = \frac{w}{\left[1 - \sum_{i=1}^{r}\left(\alpha_i + \beta_i + \frac{\gamma_i}{2}\right)\right]} \qquad \dots(2.16)$$

Proposition 2.1:

The non-zero singular point of GJR-GARCH model is stable if and only if

$$\sum_{i=1}^{r}\left(\alpha_i + \beta_i + \frac{\gamma_i}{2}\right) < 1$$

Proof:

In the neighborhood of a non-zero singular point of GJR-GARCH model with sufficiently small radius ξ_t such that $\xi_t^n \to 0$ for $n \geq 2$ we replacing $\xi + \xi_{t-i}$ instead of σ_{t-i}^2 for $i = 0,1,2,\dots,r$ in equation (2.14) we get

$\xi + \xi_t = w + \sum_{i=1}^{r}\left(\alpha_i + \beta_i + \frac{\gamma_i}{2}\right)(\xi + \xi_{t-i})$

$\xi + \xi_t = w + \sum_{i=1}^{r}\left(\alpha_i + \beta_i + \frac{\gamma_i}{2}\right)\xi + \sum_{i=1}^{r}\left(\alpha_i + \beta_i + \frac{\gamma_i}{2}\right)\xi_{t-i}$

$\therefore \xi\left(1 - \sum_{i=1}^{r}\left(\alpha_i + \beta_i + \frac{\gamma_i}{2}\right)\right) - w + \xi_t = \sum_{i=1}^{r}\left(\alpha_i + \beta_i + \frac{\gamma_i}{2}\right)\xi_{t-i}$

but $\xi\left(1 - \sum_{i=1}^{r}\left(\alpha_i + \beta_i + \frac{\gamma_i}{2}\right)\right) = w$ from (2.16)

$$\therefore\quad \xi_t = \sum_{i=1}^{r}\left(\alpha_i + \beta_i + \frac{\gamma_i}{2}\right)\xi_{t-i} \qquad \dots(2.17)$$

Equation (2.17) is a linear difference equation with constant coefficient and the characteristic equation of (2.17) can be written as

$$\lambda^{r} - \sum_{i=1}^{r}\left(\alpha_i + \beta_i + \frac{\gamma_i}{2}\right)\lambda^{r-i} = 0 \qquad \dots(2.18)$$

Then the non-zero singular point is stable if the roots of (2.18) lies inside the unite circle, i.e. $|\varphi_i| < 1$ for $i = 0,1,2,\dots,r$. Where φ_i is the root of characteristic equation for (2.18)

from (2.18)

$$\lambda^r\left(1 - \sum_{i=1}^{r}\left(\alpha_i + \beta_i + \frac{\gamma_i}{2}\right)\lambda^{-i}\right) = 0$$

$$P\left(\frac{1}{\lambda}\right) = 1 - \sum_{i=1}^{r}\left(\alpha_i + \beta_i + \frac{\gamma_i}{2}\right)\left(\frac{1}{\lambda}\right)^i = 0 \quad (\text{since } \lambda^r \neq 0) \qquad \dots(2.19)$$

then by Lemma(2.1) the polynomial (2.19) does not have a roots inside and on the unit cycle if and only if $P\left(\frac{1}{\lambda}\right) > 0$

$\because \left|\frac{1}{\lambda_i}\right| > 1$ for $i = 0,1,2,\dots,r$

$\therefore |\lambda_i| < 1$ for $i = 0,1,2,\dots,r$

and since $P(1) > 0$ then

$\left[1 - \sum_{i=1}^{r}\left(\alpha_i + \beta_i + \frac{\gamma_i}{2}\right)\right] > 0$

which is implies that $\sum_{i=1}^{r}\left(\alpha_i + \beta_i + \frac{\gamma_i}{2}\right) < 1$ ■.

Proposition 2.2:

If the GJR-GARCH(1,1) model possess a limit cycle of period $k > 0$ then this limit cycle is orbitally stable if

$$\left|\prod_{j=1}^{k}\left[\frac{\sigma_{t+j-1}}{\sigma_{t+j}}\right]\right| < \frac{1}{\left|\alpha_1 + \frac{\gamma_1}{2} + \beta_1\right|^k}$$

Proof:

Suppose that the model possess a limit cycle of period k namely

$\sigma_t^2, \sigma_{t+1}^2, \sigma_{t+2}^2, \dots, \sigma_{t+k}^2 = \sigma_t^2$

Near the neighborhood of each point of a limit cycle with sufficient small radius ξ_t such that $\xi_t^n \to 0$ for $n \geq 2$ placed $\sigma_t = \sigma_t + \xi_t$, $\sigma_{t-1} = \sigma_{t-1} + \xi_{t-1}$

The GJR-GARCH model given by :

$\sigma_t^2 = w + \alpha_1 x_{t-1}^2 + \gamma_1 I_{x_{t-1}} x_{t-1}^2 + \beta_1\,\sigma_{t-1}^2$

By taking the conditional expectation of both sides with respect to filters F_t, F_{t-1} we get:

$$\sigma_t^2 = w + \alpha_1\sigma_{t-1}^2 + \frac{\gamma_1}{2}\sigma_{t-1}^2 + \beta_1\,\sigma_{t-1}^2$$

$$\sigma_t^2 = w + \left(\alpha_1 + \frac{\gamma_1}{2} + \beta_1\right)\sigma_{t-1}^2$$

$$(\sigma_t + \xi_t)^2 = w + \left(\alpha_1 + \frac{\gamma_1}{2} + \beta_1\right)(\sigma_{t-1} + \xi_{t-1})^2$$

$$\sigma_t^2 + \xi_t^2 + 2\sigma_t\xi_t = w + \left(\alpha_1 + \frac{\gamma_1}{2} + \beta_1\right)(\sigma_{t-1}^2 + \xi_{t-1}^2 + 2\sigma_{t-1}\xi_{t-1})$$

By our assuming $\xi_t^2, \xi_{t-1}^2 \to 0$

$$\sigma_t^2 + 2\sigma_t\xi_t = w + \left(\alpha_1 + \frac{\gamma_1}{2} + \beta_1\right)(\sigma_{t-1}^2 + 2\sigma_{t-1}\xi_{t-1})$$

$$\sigma_t^2 + 2\sigma_t\xi_t = w + \left(\alpha_1 + \frac{\gamma_1}{2} + \beta_1\right)(\sigma_t^2 + 2\sigma_{t-1}\xi_{t-1})$$

$$\sigma_t^2 + 2\sigma_t\xi_t = w + \left(\alpha_1 + \frac{\gamma_1}{2} + \beta_1\right)\sigma_t^2 + \left(\alpha_1 + \frac{\gamma_1}{2} + \beta_1\right)2\sigma_{t-1}\xi_{t-1}$$

$$\sigma_t^2\left[1-\left(\alpha_1+\frac{\gamma_1}{2}+\beta_1\right)\right]-w+2\sigma_t\xi_t$$
$$=\left(\alpha_1+\frac{\gamma_1}{2}+\beta_1\right)2\sigma_{t-1}\xi_{t-1}$$

But $w=\sigma_t^2\left[1-\left(\alpha_1+\frac{\gamma_1}{2}+\beta_1\right)\right]$ we get :

$$2\sigma_t\xi_t=\left(\alpha_1+\frac{\gamma_1}{2}+\beta_1\right)2\sigma_{t-1}\xi_{t-1}$$

$$\xi_t=\frac{\left(\alpha_1+\frac{\gamma_1}{2}+\beta_1\right)}{\sigma_t}\sigma_{t-1}\xi_{t-1} \qquad \ldots(2.20)$$

From equation (2.20) and after k times we get:

$$\xi_t=\left[\frac{\left(\alpha_1+\frac{\gamma_1}{2}+\beta_1\right)}{\sigma_{t-k}}\sigma_{t-k-1}\right]\xi_{t-k}$$

But

$$\xi_{t-k}=\left[\frac{\left(\alpha_1+\frac{\gamma_1}{2}+\beta_1\right)}{\sigma_{t-k}}\sigma_{t-(k+1)}\right]\xi_{t-(k+1)}$$

$$\xi_{t+k}=\prod_{j=1}^{k}\left[\frac{\left(\alpha_1+\frac{\gamma_1}{2}+\beta_1\right)}{\sigma_{t+j}}\sigma_{t+j-1}\right]\xi_t$$

$$\therefore\ \frac{\xi_{t+k}}{\xi_t}=\prod_{j=1}^{k}\left[\frac{\left(\alpha_1+\frac{\gamma_1}{2}+\beta_1\right)}{\sigma_{t+j}}\sigma_{t+j-1}\right]$$

The equation (2.20) is a linear difference equation with a variable coefficients of the first order and its solution is a very difficult process but what interests us is that solution to the difference equation converge to zero as t gets larger $t\to\infty$, this solution is convergence, then this limit cycle is orbitally stable. This convergence only takes place if and only if the following condition is holding :

$$\left|\frac{\xi_{t+k}}{\xi_t}\right|<1 \qquad \ldots(2.21)$$

From condition (2.21) and equation (2.20) the GJR-GARCH(1,1) is orbitally stable if :

$$\left|\frac{\xi_{t+k}}{\xi_t}\right|=\left|\prod_{j=1}^{k}\left[\frac{\left(\alpha_1+\frac{\gamma_1}{2}+\beta_1\right)}{\sigma_{t+j}}\sigma_{t+j-1}\right]\right|<1$$

$$\left|\alpha_1+\frac{\gamma_1}{2}+\beta_1\right|^k\left|\prod_{j=1}^{k}\left[\frac{\sigma_{t+j-1}}{\sigma_{t+j}}\right]\right|<1$$

$$\therefore\ \left|\prod_{j=1}^{k}\left[\frac{\sigma_{t+j-1}}{\sigma_{t+j}}\right]\right|<\frac{1}{\left|\alpha_1+\frac{\gamma_1}{2}+\beta_1\right|^k}\ .\ \blacksquare$$

The GJR-GARCH model has no limit cycle.

3. Application

3.1 Data Description

We apply the stability condition of GJR-GARCH model to the data that represent the Monthly Brent Crude oil prices at closing in Dollars for the period from January 1989 to December 2018 by 359 observations obtained from the website of (https://sa.investing.com/commodities/brent-oil-historical-data).

It is worth noting that the Financial Data and Economic indicators Data are inherently unstable, so forecasts have high errors and require us to make some transforms for the purpose of obtaining numerical stability .

3.2 Modeling and Creating the GJR-GARCH model

We will apply a GJR-GARCH model to the Data series and observe and verify that forecasted conditional variance approaches the value of unconditional variance, in addition to steps to detect the heteroscedasticity, adjust the model, estimate parameters, check it's fitting, and then forecast the conditional variance .

The program used in the process of creating and programming the time series is the (MATLAB R2020a) software and the programming method has been put in an appendix with (m.file) a form at the end of the study.

3.3 Data Analysis

First step, we enter data to create a time series and we plot it, as figure (3-1) represe -nts the time series of monthly of Historical contract data for Brent Crude oil closed from: JAN. 1989 - DES. 2018.

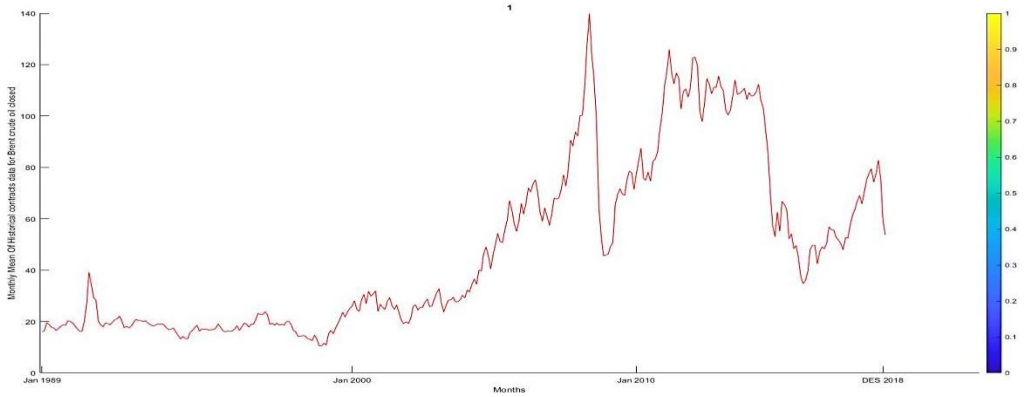

Figure (3-1) Time Series Plot Of Monthly Mean Of Historical contracts data for Brent crude oil closed From: JAN. 1989 - DES. 2018

After that, we transform the original series to the Returns series, and this is done by using the transformation known as the formula :

$$r_t=\log\frac{p_t}{p_{t-1}}$$

where r_t represent the Returns series and p_t, p_{t-1} represent the observed data at $t, t-1$ respectively, an instruction was used in MATLAB for this conversion is $r_t = price2ret(p_t)$, and figure (3-2) represent the Returns series. It suffers from some volatility and fluctuation at some values. For this reason, GJR-GARCH are useful model to analyze the fluctuations that accompany these phenomena, figure (3-3) represents the autocorrelation and partial autocorrelation functions of the data under study and the volatility at some values are appear out of the conference interval with boundaries $\pm\frac{1.96}{\sqrt{N}}$ where N is the simple size[8].

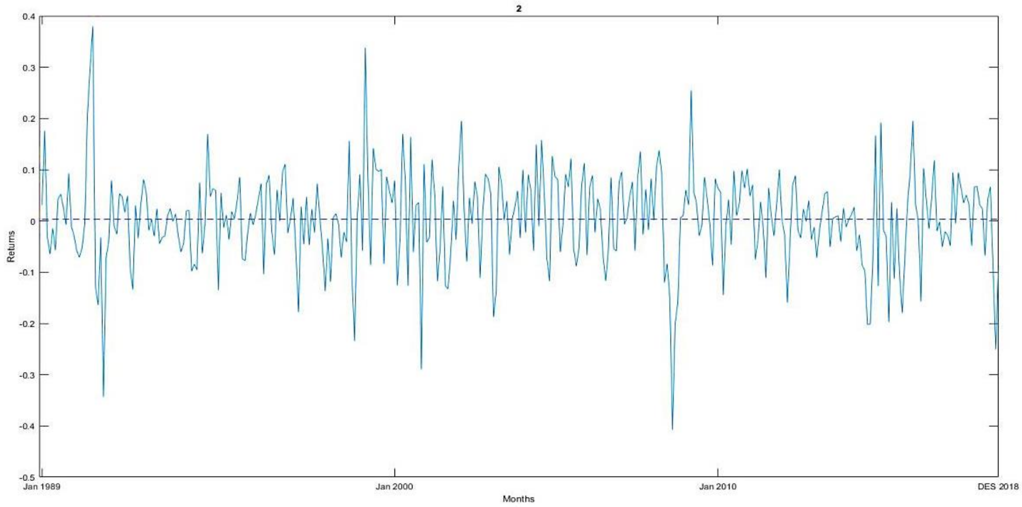

Figure (3-2) The Returns Series Of Monthly Mean Of Historical contracts data for Brent crude oil closed

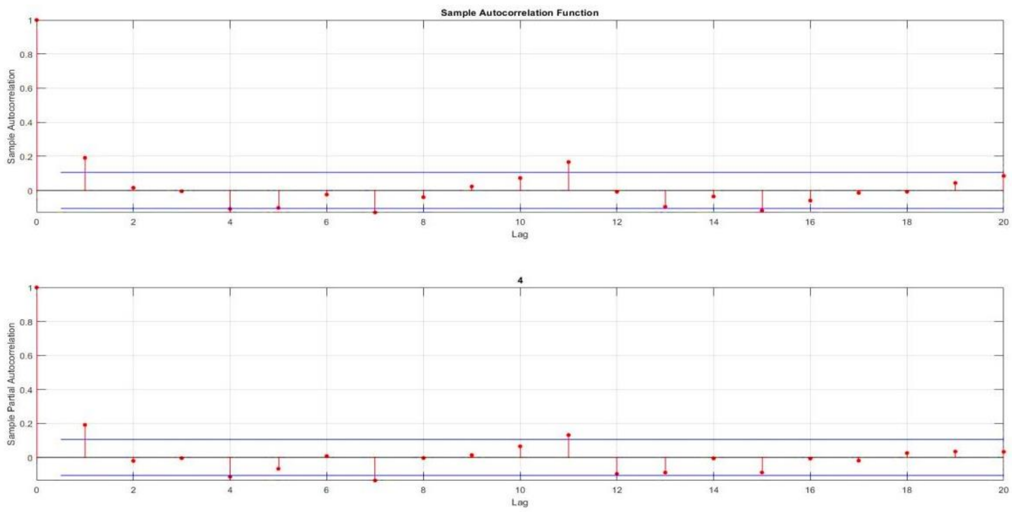

Figure (3-3) the autocorrelation and partial autocorrelation functions

Second step, it is the procedure to detect the presence of the effect of the heteroscedastic -ity variance by finding the series of Error squares for the returns series shown by the relationship $e_t = (r_t - \overline{r}_t)^2$ where $\overline{r}_t$ is the returns mean, and then drawing the autocorrelation and partial autocorrelation functions shown in the figure (3-4).

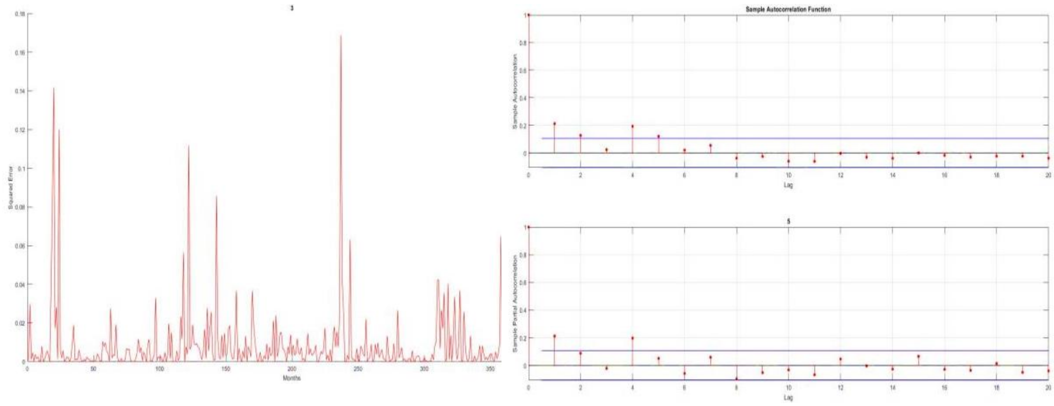

Figure (3-4) square Error Return , autocorrelation and partial autocorrelation functions

It's clear that the ACF still lie outside the boundaries of conference interval at lags $k = 1,2,4,5$ also PACF at lags $k = 1,4$.

Third step, we perform a Ljung-Box test on the return series to detect the of the heteroscedasticity effect and for 20 lags, through which it is evident as a result of the test suffering $h = 1$ from the presence of the heteroscedasticity effect as shown in the table (3-1) .

Table (3-1) results a Ljung-Box test on the return series

Lags	$h-value$	$p-value$	$Q-test$	$critical\ value$
Lag1	1	0.0003	13.2216	3.8415
Lag2	1	0.0013	13.3130	5.9915
Lag3	1	0.0040	13.3223	7.8147
Lag4	1	0.0015	17.5964	9.4877
Lag5	1	0.0007	21.4240	11.0705
Lag6	1	0.0014	21.6414	12.5916
Lag7	1	0.0002	27.7876	14.0671
Lag8	1	0.0004	28.3725	15.5073
Lag9	1	0.0008	28.5685	16.9190
Lag10	1	0.0007	30.5346	18.3070
Lag11	1	0.0000	40.8875	19.6751
Lag12	1	0.0001	40.9109	21.0261
Lag13	1	0.0000	44.3757	22.3620
Lag14	1	0.0000	44.8436	23.6848
Lag15	1	0.0000	50.0801	24.9958
Lag16	1	0.0000	51.4142	26.2962
Lag17	1	0.0000	51.4941	27.5871
Lag18	1	0.0000	51.5162	28.8693
Lag19	1	0.0001	52.2565	30.1435
Lag20	1	0.0000	55.0477	31.4104

Forth step, we fitting and estimating the parameters of the GJR-GARCH model, and the best way to estimate the parameters for the family GARCH model is by using the maximum likelihood estimation function, where this step is to choose a best order for the model by using the AIC and BIC information criteria.

Table (3-2) the value of parameters model, AIC and BIC of deferent rank from GJR-GARCH model

GJR-GARCH(Q,P)	w	α_i				γ_i				β_i				AIC	BIC
		α_1	α_2	α_3	α_4	γ_1	γ_2	γ_3	γ_4	β_1	β_2	β_3	β_4		
GJR-GARCH(0,1)	0.0059633	0.17358	-	-	-	0.33891	-	-	-	-	-	-	-	-714.0934	-702.4435
GJR-GARCH(0,2)	0.0051979	0.16122	0.13914	-	-	0.2593	0.010963	-	-	-	-	-	-	-714.0272	-694.6106
GJR-GARCH(0,3)	0.0047973	0.17315	0.15398	0.072593	-	0.25335	0.013717	-0.072593	-	-	-	-	-	-711.3425	-684.1593
GJR-GARCH(0,4)	0.0037506	0.17674	0.10047	0.079924	0.16919	0.26946	-0.077619	-0.079924	0.053192	-	-	-	-	-726.1463	-691.1964
GJR-GARCH(1,1)	0.00082587	0.22417	-	-	-	0.089064	-	-	-	0.66573	-	-	-	-732.1282	-716.5949
GJR-GARCH(1,2)	0.00084419	0.18556	0.068924	-	-	0.13663	-0.068924	-	-	0.64725	-	-	-	-728.6205	-705.3206
GJR-GARCH(1,3)	0.0010165	0.18404	0.059828	0.060669	-	0.12086	-0.059828	0.037062	-	0.57207	-	-	-	-725.1584	-694.0918
GJR-GARCH(1,4)	0.0019196	0.17972	0.051995	0.083538	0.1177	0.20508	-0.051995	-0.083538	0.19222	0.2791	-	-	-	-728.9338	-690.1005
GJR-GARCH(2,1)	0.00097472	0.2511	-	-	-	0.12756	-	-	-	0.42237	0.18512	-	-	-730.6865	-711.2699
GJR-GARCH(2,2)	0.0010628	0.19644	0.12037	-	-	0.21887	-0.12037	-	-	0.28794	0.26461	-	-	-727.9780	-700.7948
GJR-GARCH(2,3)	0.0012597	0.19244	0.15492	0.065195	-	0.22864	-0.045203	-0.065195	-	0	0.44165	-	-	-723.8722	-688.9223
GJR-GARCH(2,4)	0.0019189	0.18021	0.053352	0.08374	0.11801	0.20734	-0.053352	-0.08374	0.18881	0.27062	0.0073965	-	-	-726.9387	-684.2222
GJR-GARCH(3,1)	0.0017262	0.29567	-	-	-	0.25099	-	-	-	0.062821	0	0.35601	-	-736.2223	-712.9224
GJR-GARCH(3,2)	0.0016938	0.26125	0.11123	-	-	0.25728	-0.088731	-	-	0	0	0.3854	-	-734.2509	-703.1843
GJR-GARCH(3,3)	0.0016248	0.26619	0.10754	0.036794	-	0.25669	-0.074095	-0.036794	-	0	0	0.37274	-	-730.4638	-691.6306
GJR-GARCH(3,4)	0.00204	0.19777	0.11597	0.058474	0.13584	0.31871	-0.081361	-0.058474	0.028152	0	0	0.2165	-	-729.1261	-682.5262
GJR-GARCH(4,1)	0.0017262	0.29566	-	-	-	0.25099	-	-	-	0.062821	0	0.35601	2exp(-12)	-734.2223	-707.0391
GJR-GARCH(4,2)	0.0016938	0.26125	0.11123	-	-	0.25728	-0.08873	-	-	0	0	0.3854	2exp(-12)	-732.2509	-697.3010
GJR-GARCH(4,3)	0.0016248	0.26619	0.10754	0.036794	-	0.25669	-0.074096	-0.036794	-	0	0	0.37274	2exp(-12)	-728.4638	-685.7473
GJR-GARCH(4,4)	0.00204	0.19777	0.11597	0.058475	0.13584	0.31871	-0.081361	-0.058475	0.028153	0	0	0.2165	2exp(-12)	-727.1261	-676.6429

From table (3-2) we get the best model with less value of AIC and BIC is GJR-GARCH(3,1) model, and become the formula for the model is :

$$x_t = \sigma_t z_t \quad where\ z_t \sim iid\ N(0,1)$$

$$\sigma_t^2 = \frac{0.0017262}{(70717 \times 10^{-8})} + \left(\frac{0.29567}{(0.08774)} + \frac{0.25099}{(0.12159)} I_{x_{t-1}}\right) x_{t-1}^2 + \frac{0.062821}{(0.11321)} \sigma_{t-1}^2 + \frac{0.35601}{(0.096002)} \sigma_{t-3}^2 \quad \dots (3.1)$$

where $I_{x_{t-i}}$ is indicator function defined as

$$I_{x_{t-i}} = \begin{cases} 0\ if\ x_{t-i} < 0 \\ 1\ if\ x_{t-i} \geq 0 \end{cases} \quad \forall i \quad \dots(3.2)$$

By applying the stability conditions to the model, we find that model is stable according to the condition (2.10) then

$$\alpha_1 + \frac{\gamma_1}{2} + \sum_{j=1}^{3} \beta_j = 0.29567 + \frac{0.25099}{2} + 0.062821 + 0.35601 = 0.839996 < 1$$

And the value of unconditional variance for GJR-GARCH model given in equation (2.15) is

$$\sigma^2 = \frac{w}{1-\alpha_1+\frac{\gamma_1}{2}+\sum_{j=1}^{3}\beta_j} = \frac{0.0017262}{1-0.839996} = 0.01078848 \approx 0.0108$$

Fifth step, this step complete in two stages. The first stage is finding and calculating the standardized residual series of the model through the equation: $\hat{r}_t = \frac{r_t}{\sigma_t}$ where $\hat{r}_t$ is standard residual series, r_t is white noise residual for return series and σ_t is standard deviation. Then draw the square standardized residual series, draw the normal distribution line of standardized residual series with the normal distribution line and draw the autocorrelation and partial autocorrelation functions of square standardized residual series. The second stage is perform a Ljung-Box test on its series. It can be observed in figure (3-5) and table (3-3).

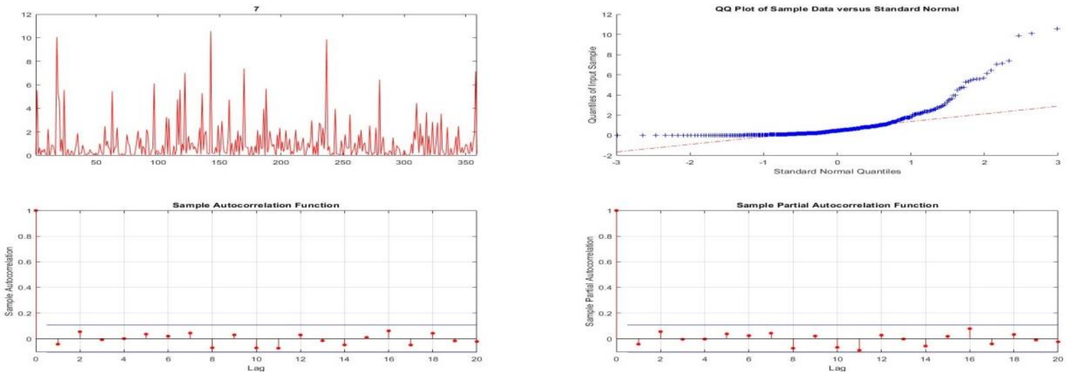

Figure (3-5) square standardized residual, distribution curve and autocorrelation and partial autocorrelation functions of square standardized residual series

Table (3-3) Ljung-Box test for square standardized residual series

Lags	$h-value$	$p-value$	$Q-test$	$critical\ value$
Lag1	0	0.4142	0.6666	3.8415
Lag2	0	0.4224	1.7235	5.9915
Lag3	0	0.6259	1.7500	7.8147
Lag4	0	0.7816	1.7501	9.4877
Lag5	0	0.8203	2.2037	11.0705
Lag6	0	0.8853	2.3461	12.5916
Lag7	0	0.8803	3.0507	14.0671
Lag8	0	0.7649	4.9313	15.5073
Lag9	0	0.8113	5.2573	16.9190
Lag10	0	0.7073	7.1915	18.3070
Lag11	0	0.5983	9.2560	19.6751
Lag12	0	0.6531	9.5761	21.0261
Lag13	0	0.7207	9.6697	22.3620
Lag14	0	0.7201	10.5611	23.6848
Lag15	0	0.7807	10.5953	24.9958
Lag16	0	0.7441	11.9979	26.2962
Lag17	0	0.7407	12.9326	27.5871
Lag18	0	0.7537	13.6166	28.8693
Lag19	0	0.7992	13.7307	30.1435
Lag20	0	0.8345	13.9206	31.4104

From the table (3-3), we notice that the Ljung-Box test results for the residual series of the GJR-GARCH(3,1) model are not correlated, and a heteroscedasticity has been removed. Therefore, the variance equation for GJR-GARCH(3,1) model is fitting, and thus the suitability of the model is verified.

Sixth step, the last step we inferred and the forecasted conditional variance for the series, we get the conditional variance converge to unconditional variance.

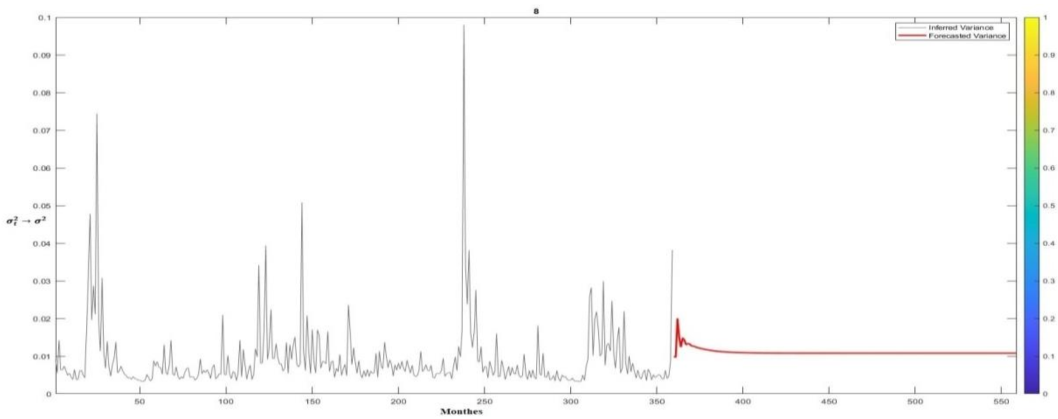

Figure (3-6) the inferred and the forecasted conditional variance for the series

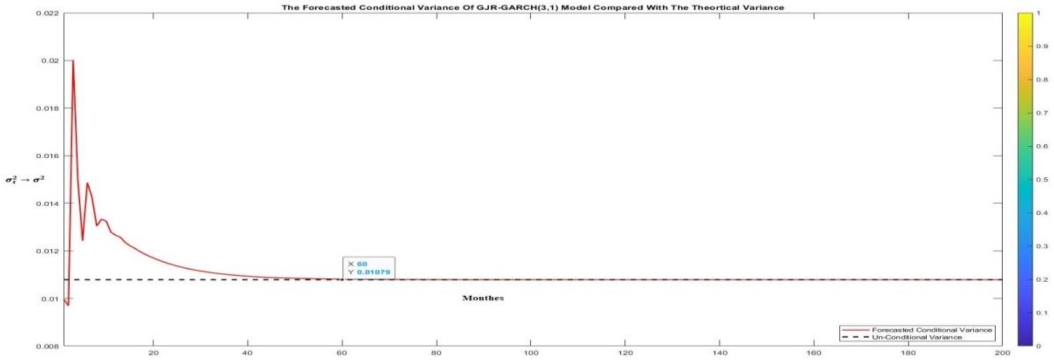

Figure (3-7) the conditional variance converge to unconditional variance

Appendix 1

```
Data = [15.91 ;16.40 ;19.55 ;18.94 ;17.75 ;17.49 ;16.52 ;17.24 ;18.16 ;18.63 ;18.49 ;20.28 ;
20.06 ;19.48 ;18.38 ;17.12 ;16.24 ;16.14 ;19.84 ;26.75 ;39.10 ;34.41 ;29.20 ;28.27 ;
20.06 ;18.68 ;17.95 ;19.42 ;19.21 ;18.72 ;19.74 ;20.70 ;21.05 ;22.10 ;20.13 ;17.61 ;
18.15 ;17.55 ;18.14 ;19.66 ;20.79 ;20.41 ;20.47 ;19.87 ;20.34 ;19.45 ;18.84 ;18.29 ;
18.47 ;18.92 ;18.90 ;19.15 ;18.60 ;17.51 ;16.75 ;17.08 ;17.43 ;15.80 ;14.52 ;13.20 ;
14.22 ;13.35 ;13.25 ;15.69 ;16.45 ;17.52 ;18.59 ;16.24 ;17.15 ;16.92 ;17.11 ;16.50 ;
16.80 ;16.87 ;17.50 ;19.06 ;17.70 ;16.38 ;16.01 ;16.25 ;16.12 ;16.33 ;17.04 ;18.33 ;
16.52 ;17.76 ;19.41 ;19.02 ;17.80 ;18.91 ;18.90 ;20.78 ;23.21 ;22.67 ;22.77 ;23.81 ;
22.52 ;18.85 ;19.38 ;18.52 ;19.40 ;18.51 ;18.94 ;18.51 ;19.90 ;20.02 ;18.94 ;16.52 ;
15.96 ;14.17 ;14.26 ;14.46 ;14.37 ;13.38 ;13.09 ;12.56 ;14.68 ;13.22 ;10.46 ;10.53 ;
11.53 ;10.88 ;15.25 ;16.57 ;15.20 ;17.51 ;19.37 ;21.33 ;23.58 ;21.69 ;23.64 ;25.08 ;
25.97 ;28.09 ;24.77 ;23.89 ;28.31 ;30.57 ;26.93 ;31.72 ;29.84 ;30.76 ;31.88 ;23.87 ;
26.66 ;25.57 ;24.74 ;27.89 ;29.34 ;26.08 ;24.69 ;26.41 ;23.26 ;20.37 ;19.14 ;19.90 ;
19.18 ;21.33 ;25.92 ;26.47 ;24.45 ;25.58 ;25.44 ;27.47 ;28.75 ;25.72 ;26.16 ;28.66 ;
31.10 ;32.79 ;27.18 ;23.68 ;26.32 ;28.33 ;28.37 ;29.49 ;27.61 ;27.70 ;28.45 ;30.17 ;
29.18 ;32.23 ;31.51 ;34.48 ;36.58 ;34.50 ;40.03 ;39.61 ;46.38 ;48.98 ;45.51 ;40.46 ;
45.92 ;50.06 ;54.29 ;51.09 ;50.73 ;55.58 ;59.37 ;67.02 ;63.48 ;58.10 ;55.05 ;58.98 ;
65.99 ;61.76 ;65.91 ;72.02 ;70.41 ;73.51 ;75.15 ;70.25 ;62.48 ;59.03 ;64.26 ;60.86 ;
57.40 ;61.89 ;68.10 ;67.65 ;68.04 ;71.41 ;77.05 ;72.69 ;79.17 ;90.63 ;88.26 ;93.85 ;
92.21 ;100.10 ;100.30 ;111.36 ;127.78 ;139.83 ;123.98 ;114.05 ;98.17 ;65.32 ;53.49 ;45.59 ;
45.88 ;46.35 ;49.23 ;50.80 ;65.52 ;69.30 ;71.70 ;69.65 ;69.07 ;75.20 ;78.47 ;77.93 ;
71.46 ;77.59 ;82.70 ;87.44 ;75.65 ;75.01 ;78.18 ;74.64 ;82.31 ;83.15 ;85.92 ;94.75 ;
101.01 ;111.80 ;117.36 ;125.89 ;116.73 ;112.48 ;116.74 ;114.85 ;102.76 ;109.56 ;110.52 ;107.38 ;
110.98 ;122.66 ;122.88 ;119.47 ;101.87 ;97.80 ;104.92 ;114.57 ;112.39 ;108.70 ;111.23 ;111.11 ;
115.55 ;111.38 ;110.02 ;102.37 ;100.39 ;102.16 ;107.70 ;114.01 ;108.37 ;108.84 ;109.69 ;110.80 ;
106.40 ;109.07 ;107.76 ;108.07 ;109.41 ;112.36 ;106.02 ;103.19 ;94.67 ;85.86 ;70.15 ;57.33 ;
52.99 ;62.58 ;55.11 ;66.78 ;65.56 ;63.59 ;52.21 ;54.15 ;48.37 ;49.56 ;44.61 ;37.28 ;
34.74 ;35.97 ;39.60 ;48.13 ;49.69 ;49.68 ;42.46 ;47.04 ;49.06 ;48.30 ;50.47 ;56.82 ;
55.70 ;55.59 ;52.83 ;51.73 ;50.31 ;47.92 ;52.65 ;52.38 ;57.54 ;61.37 ;63.57 ;66.87 ;
69.05 ;65.78 ;70.27 ;75.17 ;77.59 ;79.44 ;74.25 ;77.42 ;82.72 ;75.47 ;58.71 ;53.80 ];
figure(1)
hold on
xlabel('Months');
h = gca;
h.XTick = [1 133 254 359];
h.XTickLabel = {'Jan 1989','Jan 2000','Jan 2010',...  'DES 2018'};
ylabel('Monthly Mean Of Historical contracts data for Brent crude oil closed');
title('Time Series Plot Of Monthly Mean Of Historical contracts data for Brent crude oil closed From: JAN.1990
- DES.2018')
plot(Data,'r');
hold off
r=price2ret(Data);
N =length(r);
meanR = mean(r);
error = r - mean(r);
squerror = error.^2;
```

```
figure(2)
plot(r)
hold on
plot(meanR*ones(N,1),'--b')
xlim([0,N])
xlabel('Months');
h = gca;
h.XTick = [1 133 254 359];
h.XTickLabel = {'Jan 1989','Jan 2000','Jan 2010',...   'DES 2018'};
ylabel('Returns');
title('Plot The Returns Series Of Monthly Mean Of Historical contracts data for Brent crude oil closed')
hold off
figure(3)
hold on
xlabel('Months');ylabel('Squared Error');
title('Plot Of Squared Errors Return')
plot(squerror,'r');
hold off
figure(4)
subplot(2,1,1)
autocorr(r)
subplot(2,1,2)
parcorr(r)
title('Partial Autocorrelation Functions Of Return Series')
 figure(5)
subplot(2,1,1)
autocorr(squerror)
subplot(2,1,2)
parcorr(squerror)
title('Partial Autocorrelation Functions Of Squared Errors Return Series')
[h,pValue,Qstat,cValue] = lbqtest(r,'Lags',[1:20])
Q=3;
P=1;
Mdl=gjr(Q,P);
[EstMdl,EstParamCov,LogL,info] = estimate(Mdl,r);
numParams = sum(any(EstParamCov));
[AIC,BIC] = aicbic(LogL,numParams,N)
rng default;
V=infer(EstMdl,r);
figure(6)
plot(V,'r')
xlim([1,N])
h = gca;
h.XTick = [1 133 254 359];
h.XTickLabel = {'Jan 1989','Jan 2000','Jan 2010',...
    'DES 2018'};
title('Infered Conditional Variance')
StdRes=r./sqrt(V);
SquStdRes=StdRes.^2;
figure(7)
subplot(2,2,1)
plot(SquStdRes,'r')
xlim([1,N])
title('Squared Standardized Residuals')
subplot(2,2,2)
qqplot(SquStdRes)
subplot(2,2,3)
autocorr(SquStdRes)
subplot(2,2,4)
parcorr(SquStdRes)
```

```
[h,pValue,Qstat,cValue]=lbqtest(SquStdRes,'lags',[1:20])
Vf = forecast(EstMdl,200,'Y0',r,'V0',V);
figure(8)
plot(V,'Color',[.4,.4,.4])
hold on
plot(N+1:N+200,Vf,'r','LineWidth',2)
xlim([1,N+200])
legend('Inferred Variance','Forecasted Variance','Location','Northwest')
title('The Inferred And The Forecasted Conditional Variance For The Series Of Monthly Mean Of Historical contracts data for Brent crude oil closed')
hold off
UnConVar=EstMdl.Constant/(1-EstMdl.GARCH{1}-EstMdl.GARCH{2}-EstMdl.GARCH{3}-
EstMdl.ARCH{1}-0.5.*EstMdl.Leverage{1})
rng default;
Vsim = simulate(EstMdl,200,'NumPaths',1000,'E0',r,'V0',V);
sim = mean(Vsim,2);
figure(9)
plot(Vf,'r','LineWidth',1.5)
hold on
plot(ones(200,1)*UnConVar,'k--','LineWidth',2)
xlim([1 200]);
title('The Forecasted Conditional Variance Of GJR-GARCH(3,1) Model Compared With The Theortical Variance')
legend('Forecasted Conditional Variance','Un-Conditional Variance','Location','southEast')
hold off
```

References

[1] Bollerslv, T. (1986). Generalized autoregressive conditional heteroskedasticity. *journal of Econometrics*, **31(3):**307-327.

[2] Engle, R.F. (1982). Autoregressive Conditional Heteroscedasticity with Estimates variance of United Kingdom Inflation. *journal of Econometrica*, **50 (4)**: 987–1008.

[3] Francq, C. and Zakoian, J. M. (2019).GARCH Models: Structure. Statistical Inference and Financial Applications.2nd edn., John Wiley and Ltd Publication:492 pp.

[4] GLOSTEN, L.R. ; JAGANNATHAN, R. and RUNKLE, D.E. (1993). On the Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks. *The Journal of Finance*, **485 (5)**: 1779–1801.

[5] Mohammad , A.A. and Salim, A.J. (2007). Stability of Logistic Autoregressive model. *Qatar Univesity of scince journal*, **27**:17–28.

[6] Mohammad, A.A. and Gannam, A.K. (2010). Stability of Cauchy Autoregressive model. *journal of pure and applied scince, Salahaddin University Hawler* **(special Issue):**52-62.

[7] Mohammad, A.A. and Ghaffar, M.K. (2016). Astudy on stability of Conditional variance for GARCH models with application. Tikrit journal of pure scince, **21 (4):**160-169.

[8] Mohammad, A.A. and Mudhir, A.A. (2018). Dynamical approach in studying stability condition of exponential (GARCH) models. *Journal of King Saud University - Science*, Vol.**32 (1):**272-278.

[9] Mohammad, A.A. and Salim, A.J. (1996). The Analysis and Modeling of the time series of annual mean temprature in Mosul City. *Rafidain Journal of Science*,**7 (1):** 37-48.

[10] Nelson, D.B. (1991). Conditional Heteroskedasticity in Asset Returns: A New Approach", *Econometrica*, **59 (2)**:347-370.

[11] Ozaki, T. (1985). Non-linear time series models and Dynamical systems. Handbook of statestics. Hannen et al. Elsevier science publishers Rv ,**5 (S)**:25–83.

[12] Wiphatthananthakula, C. and Sriboonchittab, S. (2010). The Comparison among ARMA-GARCH, -EGARCH, -GJR. *The Thailand Econometrics Society journal*, **2 (2):**140–148.

[13] Yuan Chen, M. (2013).Time series Analysis: Conditional Volatility models. Dep. of Finance. National chung Hsing University: 1-42 pp

منهجية حركية في دراسة نماذج GJR-GARCH(Q,P) مع التطبيق

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الملخص

هذه الورقة تتعلق بإيجاد شروط استقرارية نماذج GJR-GARCH(Q,P) بأستخدام تقنية التقريب الخطي المحلية في سبيل تحويل النموذج اللاخطي الى معادلة فرقية بمعاملات ثابتة وعن طريقها يمكن ايجاد شروط الاستقرارية بدراسة جذور المعادلة المميزة لها. بالاضافة الى ذلك تم مناقشة وإيجاد شروط الاستقرارية المدارية لنموذج GJR-GARCH(1,1) عندما يمتلك دورة نهاية وبأية دورة. وأخيراً تم تطبيق شروط الاستقرارية التي تم إيجادها على بيانات حقيقية تمثل المعدل الشهري لأسعار الاغلاق بالدولار لعقود نفط خام برنت للسنوات (1989−2019) ووجدنا بأن افضل نموذج يمثل هذه البيانات هو GJR-GARCH(3,1) حسب معياري AIC و BIC للمعلوماتية.