ON SPECTRA OF SOME TENSORS OF SIX-DIMENSINAL KÄHLERIAN AND NEARLY-KÄHLERIAN SUBMANIFOLDS OF CAYLEY ALGEBRA

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Abstract

Six-dimensional Kählerian and nearly-Kählerian submanifolds of Cayley algebra are considered. Spectra of some classical tensors of such submanifolds of the octave algebra are computed. It is proved that a nearly-Kählerian six-dimensional submanifold of Cayley algebra is conharmonically flat if and only if it is holomorphically isometric to the complex Euclidean space C^3 with a canonical Kählerian structure.

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Introduction

The almost Hermitian structures (AH-structures) belong to the most substantial differen tial-geometrical structures. A great number of signification publications have been devoted to the study of such structures. These works characterize almost Hermitian structures from the point of view of differential geometry as well as of modern theoretical physics. Such well-known mathematicians as R. Brown, E. Calabi, N. Ejiri, A. Gray, L.M. Hervella, K.-T. Kim, T. Koda, M. Prvanovic, G. Rizza, K. Sekigawa, I. Vaisman, L. Vanhecke, K. Yano made a great contribution to the theory of almost Hermitian structures. No doubt that one of the first places in this list of names will be occupied by the Russian geometer Vadim Feodorovich Kirichenko who has obtained a set of important results in this field (see, for instance, [27] - [36]). He was one of the fist scientists to use systematically the method of associated G-structures. This method is the modern variant of Cartan's exterior form method [11] developed by G. F. Laptev and A.M. Vasil'ev. Before V.F. Kirichenko, most significant investigation on almost Hermitian structures was done in terms of Koszul's invariant calculation [37]. Without denying the effective Koszul's calculation system, V.F. Kirichenko and his pupils (H. Aboud, A. Abu-Saleem, M. Banaru, I. Borisovsky, A. Gritsans, A. Rustanov, A. Shihab, L. Stepanova, E. Volkova, B. Zayatuev and others) have obtained the principal part of their results precisely by the method of associated G-structures.

The existence of 3–vector cross products on Cayley algebra gives a lot of substantial examples of almost Hermitian manifolds. As it is well known, every 3-vector cross product on Cayley algebra induces a 1-vector cross product (or, what is the same in this case, an almost Hermitian structure) on its six–dimensional oriented submanifold (see [16]). Such almost Hermitian structures (in particular, nearly-Kählerian structures) were studied by a number of authors: E. Calabi [10], A. Gray [14], [15], [16], [17], V.F. Kirichenko [27], [29], {30], [32], Haizhong Li and Guoxin Wei [21], [22], H. Hashimoto [22], [23], [24], K. Sekigawa [38], L. Vranchen [40], N. Ejiri [12], S. Funabashi and J.S. Pak [13] and others. For example, a complete classification of Kählerian and nearly-Kählerian structures on six-dimensional submanifolds of the octave algebra has been obtained [27], [29].

It is known that the classes of Kählerian and nearly-Kählerian manifolds are the most important classes of almost Hermitian manifolds. We also note that all Cray-Hervella classes of almost Hermitian manifolds include the class of Kählerian manifolds [20].

Our main result is the computation of the spectra for some classical tensors. We have obtained all types of the components for the tensor of Riemannian curvature, the Ricci tensor, the Weyl tensor of conformal curvature and the conharmonic curvature tensor.

From above mentioned results and from our previous results [3], [5], [6], [8], [9], [36], [39] we deduce some conclusions on geometry of Kählerian and nearly-Kählerian submanifolds of Cayley algebra.

Preliminaries

We consider an almost Hermitian manifold, i.e. a $2n$-dimensional manifold, having a Riemannian metric $g=\langle\cdot,\cdot\rangle$ and an almost complex structure J. Besides the following condition must hold

$\langle JX, JY\rangle = \langle X, Y\rangle, \quad \forall X, Y \in \aleph(M^{2n})$,

where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n} [20], [35]. All manifolds, tensor fields and similar objects are assumed to be of the class C^{∞}.

The specification of an almost Hermitian structure on a manifold is equivalent to the setting of a G-structure, where G is the unitary group $U(n)$ [31], [35]. Its elements are the frames adapted to the structure (A-frames). They look as follows:

$(p, \varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{\hat{1}}, \ldots, \varepsilon_{\hat{n}})$

where ε_a are the eigenvectors corresponded to the eigenvalue $i=\sqrt{-1}$, and $\varepsilon_{\hat{a}}$ are the eigenvectors corresponded to the eigenvalue $-i$. Here and further the index a ranges from 1 to n, and we state $\hat{a}=a+n$.

It is reasonable to consider the most important tensor written in an A-frame. This corresponds to the problems of the study of almost Hermitian manifolds. V.F. Kirichenko, who worked out such a method [31], has introduced the notion of the tensor spectrum.

The spectra of the tensors, determining the almost Hermitian structure on a manifold, look as follows [31], [35]:

1) $g_{ab}=0, \quad g_{\hat{a}b}=\delta_b^a, \quad g_{a\hat{b}}=\delta_a^b, \quad g_{\hat{a}\hat{b}}=0;$

2) $J_b^a=i\delta_b^a, \quad J_b^{\hat{a}}=0, \quad J_{\hat{b}}^a=0, \quad J_{\hat{b}}^{\hat{a}}=-i\delta_a^b.$

(Here and further the indices a, b, c, d, h range from 1 to n, and we set $\hat{a}=a+n$, $i=\sqrt{-1}$).

We recall that the fundamental (or Kählerian) form of an almost Hermitian manifold is determined by

$F(X,Y)=\langle X, JY\rangle, \qquad X,Y\in\aleph(M^{2n})$.

The spectrum of the fundamental form of an almost Hermitian manifold looks as follows [35]:

$F_{ab}=0, \quad F_{\hat{a}b}=i\delta_b^a, \quad F_{a\hat{b}}=-i\delta_a^b, \quad F_{\hat{a}\hat{b}}=0.$

Thus, the matrices of the Riemannian metric, of the almost complex structure J and of the fundamental form F of an almost Hermitian manifold in an A-frame will be written down as follows:

$$(g_{kj})=\left(\begin{array}{c|c}0 & I_n\\ \hline I_n & 0\end{array}\right); \quad (J_j^k)=\left(\begin{array}{c|c}iI_n & 0\\ \hline 0 & -iI_n\end{array}\right); \quad (F_{kj})=\left(\begin{array}{c|c}0 & iI_n\\ \hline -iI_n & 0\end{array}\right), \quad (1)$$

where I_n is the identity matrix.

From [18], [19], [20], [28], we note that the almost Hermitian structure is called nearly-Kählerian structure (in short, NK-structure) if its fundamental form $F(X,Y)=\langle X, JY\rangle$ is a Killing form or that is equivalent to the condition

$\nabla_X(J)X=0; \qquad X\in\aleph(M^{2n})$.

The nearly-Kählerian structure is called Kählerian if $\nabla J=0$, otherwise it is a proper nearly-Kählerian structure. It is well known [35] that the structural equations of the Riemannian connection of a nearly-Kählerian structure on the space of the corresponding G-structure (called the Cartan structural equations of the nearly-Kählerian structure) have the following form:

1. $d\omega^a=\omega_b^a\wedge\omega^b+B^{abc}\omega_b\wedge\omega_c;$
2. $d\omega_a=-\omega_a^b\wedge\omega_b+B_{abc}\omega^b\wedge\omega^c; \qquad (2)$
3. $d\omega_b^a=\omega_c^a\wedge\omega_b^c+(2B^{adh}B_{hbc}+A_{bc}^{ad})\omega^c\wedge\omega_d,$

where $\omega_a=\omega^{\hat{a}}$.

Here $\{B^{abc}\}$, $\{B_{abc}\}$ and $\{A_{bc}^{ad}\}$ are the systems of functions in the associated space of the G-structure serving the components of the complex tensors on M^{2n}. These tensors are called Kirichenko structural tensors [1] of the first, second and third order, respectively. Being so, the structural tensors of the first and second order are skew-symmetrical and the structural tensors of the third order are symmetrical by any pair of superscript or subscript indices [1].

In addition, it is known that the components of the Kirichenko structural tensors of the first and second order of a NK-manifold satisfy the following identities:

1. $dB^{abc}-B^{hbc}\omega_h^a-B^{ahc}\omega_h^b-B^{abh}\omega_h^c=0;$
2. $dB_{abc}-B_{hbc}\omega_a^h-B_{ahc}\omega_b^h-B_{abh}\omega_h^c=0. \qquad (3)$

We remark that the Kirichenko tensors were used mainly for characterization of Hermitian manifolds, i.e. of almost Hermitian manifolds with an integrable almost Hermitian structure (see, for instance, [1], [2], [3], [4], [6], [7]).

The relationships (3) are equivalent to the parallelism of these tensors in the associated connection [1]. The components of the third order Kirichenko structural tensor satisfy the following identity:

$$dA_{bc}^{ad}+A_{hc}^{ad}\omega_b^h+A_{bh}^{ad}\omega_c^h-A_{bc}^{hd}\omega_h^a-A_{bc}^{ah}\omega_h^d=A_{bch}^{ad}\omega^h+A_{bc}^{adh}\omega_h,$$

where $\{A_{bch}^{ad}\}$ and $\{A_{bc}^{adh}\}$ are the systems of functions in the space of the corresponding G-structure that are used as the components of the covariant differential structural tensor of the third order in the associated connection. We remark that they are symmetrical by any pair of superscript or subscript indices. (We can remind that the associated connection of an almost Hermitian manifold is called the connection $\tilde{\nabla}=\nabla+T$, where T is the composite tensor of the adjoint Q-algebra [31]). We can also note that the components of structural tensor of a NK-manifold satisfy the formulae of complex conjugation [35]:

$$\overline{B^{abc}}=B_{abc}; \quad \overline{A_{bc}^{ad}}=A_{ad}^{bc}. \quad (4)$$

As is known, the structural tensor of the first and second order vanish precisely when the manifold is Kählerian [3]. For convenience, we will take the following notations:

$B_{bc}^{ad}=B^{adh}B_{hbc}; \quad B_b^a=B_{bc}^{ac};$

$B=B_a^a; \quad A_b^a=A_{bc}^{ac}, \quad A=A_a^a.$

Let R, Ric, K be the Riemannian curvature tensor (or the Riemann-Christoffel tensor [11], [35]), the Ricci tensor and the scalar curvature of the manifold M^{2n}, respectively. We remind that the main nonzero components of these tensors on the space of the G-structure (i.e. in an A-frame) for a NK-manifold have the following form, respectively [35]:

$R_{\hat{a}bc\hat{d}}=-2B_{cd}^{ab}; \quad R_{\hat{a}bc\hat{d}}=B_{bc}^{ad}+A_{bc}^{ad}; \quad R_{abcd}=R_{\hat{a}bcd}=0;$

$Ric_{ab}=0; \quad Ric_{\hat{a}b}=3B_b^a-A_b^a; \qquad (5)$

$K=6B-2A.$

The other nonzero components of these tensors are calculated with regard to the reality and classical characteristics of symmetry [35].

The main results

As it is known [20], [35], a necessary and sufficient condition for an almost Hermitian structure to be nearly-Kählerian is the following:

$\nabla_X(F)(X,Y)=0.$

Using the definitions of Kirichenko tensors, by direct computing it is possible to reformulate this condition in terms of Kirichenko tensors.

THEOREM A [27], [29]. An almost Hermitian structure is nearly-Kählerian if and only if

$B^{abc}=-B^{bac}$, $B_{abc}=-B_{bac}$;

$B^{ab}{}_c=0$, $B_{ab}{}^c=0$.

That is why we can rewrite the Cartan structural equations for the case when the almost Hermitian structure is nearly-Kählerian.

THEOREM B [27], [3]. The first group of Cartan structural equations of a nearly-Kählerian structure is the following:

$$d\omega^a=\omega^a_b\wedge\omega^b + B^{abc}\,\omega_b\wedge\omega_c;$$
$$d\omega_a=-\omega^b_a\wedge\omega_b + B_{abc}\,\omega^b\wedge\omega^c, \qquad (6)$$

where $B^{abc}=-B^{bac}$, $B_{abc}=-B_{bac}$.

Let $\mathbf{O}\equiv R^8$ be the Cayley algebra. As it is well-known [16], two non-isomorphic three-fold vector cross products are defined on it by means of the relations:

$$P_1(X,Y,Z)=-X(\bar{Y}Z)+\langle X,Y\rangle Z+\langle Y,Z\rangle X-\langle Z,X\rangle Y,$$
$$P_2(X,Y,Z)=-(X\bar{Y})Z+\langle X,Y\rangle Z+\langle Y,Z\rangle X-\langle Z,X\rangle Y,$$

where $X,Y,Z\in\mathbf{O}$, $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbf{O}$ and $X\to\bar{X}$ is the conjugation operator. Moreover, any other three-fold vector cross product in the octave algebra is isomorphic to one of the above-mentioned two.

If $M^6\subset\mathbf{O}$ is a six–dimensional oriented submanifold, then the induced almost Hermitian structure $\{J_\alpha, g=\langle\cdot,\cdot\rangle\}$ is determined by the relation

$$J_\alpha(X)=P_\alpha(X,e_1,e_2),\quad \alpha=1,2,$$

where $\{e_1,e_2\}$ is an arbitrary orthonormal basis of the normal space of M^6 at a point p, $X\in T_p(M^6)$ [16].

We recall that the point $p\in M^6$ is called general [27], [29], if

$e_0\notin T_p(M^6)$,

where e_0 is the unit of Cayley algebra. A submanifold $M^6\subset\mathbf{O}$, consisting only of general points, is called a general-type submanifold [27]. In what follows, all submanifolds M^6 that will be considered are assumed to be of general type.

Let's use the Cartan structure equations of the Hermitian $M^6\subset\mathbf{O}$ obtained in [29].

$$d\omega^a=\omega^a_b\wedge\omega^b+\frac{1}{\sqrt{2}}\varepsilon^{abh}D_{hc}\omega^c\wedge\omega_b+\frac{1}{\sqrt{2}}\varepsilon^{ah[b}D_h{}^{c]}\omega_b\wedge\omega_c;$$
$$d\omega_a=-\omega^b_a\wedge\omega_b+\frac{1}{\sqrt{2}}\varepsilon_{abh}D^{hc}\omega_c\wedge\omega^b+\frac{1}{\sqrt{2}}\varepsilon_{ah[b}D^h{}_{c]}\omega^b\wedge\omega^c; \qquad (7)$$
$$d\omega^a_b=\omega^a_c\wedge\omega^c_b-\left(\frac{1}{2}\delta^{ah}_{bg}D_{h[k}D^g{}_{j]}+\sum_\varphi T^\varphi_{\hat{a}[k}T^\varphi_{j]b}\right)\omega^k\wedge\omega^j,$$

where

$D_{cj}=\mp T^8_{cj}+iT^7_{cj}$; $D_{\hat{c}j}=\mp T^8_{\hat{c}j}-iT^7_{\hat{c}j}$;

$D^{hc}=D_{\hat{h}\hat{c}}$; $D_h{}^c=D_{h\hat{c}}$; $D^h{}_c=D_{\hat{h}c}$;

$\delta^{ah}_{bg}=\delta^a_b\delta^h_g-\delta^a_g\delta^h_b$,

$\varepsilon_{abc}=\varepsilon^{123}_{abc}$, $\varepsilon^{abc}=\varepsilon^{abc}_{123}$ are the components of the third-order Kronecher tensor [29], $\{T^\varphi_{kj}\}$ are the components of the configuration tensor (or the Euler curvature tensor [11]) of $M^6\subset\mathbf{O}$.

We note that $\bar{\omega}^a=\omega_a$, $\bar{\omega}^a_b=-\omega^b_a$. Here and further

$a,b,c,d,e,f,g,h=1,2,3;\quad \psi,\varphi=7,8;$

$i,j,k,l,m,n,p,r,s=1,2,3,4,5,6;$

$\hat{a}=a+3.$

Taking into account (6), we can rewrite the Cartan structural equations for case when the almost Hermitian $M^6\subset\mathbf{O}$ is nearly-Kählerian:

$$d\omega^a=\omega^a_b\wedge\omega^b+\frac{1}{\sqrt{2}}\varepsilon^{ah[b}D_h{}^{c]}\omega_b\wedge\omega_c;$$
$$d\omega_a=-\omega^b_a\wedge\omega_b++\frac{1}{\sqrt{2}}\varepsilon_{ah[b}D^h{}_{c]}\omega^b\wedge\omega^c; \qquad (8)$$
$$d\omega^a_b=\omega^a_c\wedge\omega^c_b-\left(\frac{1}{2}\delta^{ah}_{bg}D_{h[k}D^g{}_{j]}+\sum_\varphi T^\varphi_{\hat{a}[k}T^\varphi_{j]b}\right)\omega^k\wedge\omega^j.$$

THEOREM 1. For six-dimensional almost Hermitian submanifolds of Cayley algebra the following equivalence is fulfilled:

$$B^{abc}=-B^{bac}\quad\Leftrightarrow\quad D_{d\hat{a}}=\frac{1}{3}\delta^a_d tr(D_{d\hat{a}}).$$

Proof

1. Let $B^{abc}=-B^{bac}$, i.e. $B^{abc}+B^{bac}=0$.

Now let us use the expressions for Kirichenko structural tensors for six-dimensional almost Hermitian submanifolds of Cayley algebra from (8):

$$\frac{1}{\sqrt{2}}\varepsilon^{ah[b}D_h{}^{c]}+\frac{1}{\sqrt{2}}\varepsilon^{bh[a}D_h{}^{c]}=0;$$
$$\frac{1}{\sqrt{2}}\varepsilon^{ahb}D_{h\hat{c}}-\frac{1}{\sqrt{2}}\varepsilon^{ahc}D_{h\hat{b}}+\frac{1}{\sqrt{2}}\varepsilon^{bha}D_{h\hat{c}}-\frac{1}{\sqrt{2}}\varepsilon^{bhc}D_{h\hat{a}}=0;$$
$$\varepsilon^{ahc}D_{h\hat{b}}+\varepsilon^{bhc}D_{h\hat{a}}=0;$$
$$(\varepsilon_{cfd}):\qquad \delta^{ah}_{fd}D_{h\hat{b}}+\delta^{bh}_{fd}D_{h\hat{a}}=0;$$
$$(\delta^a_f\delta^h_d-\delta^a_d\delta^h_f)D_{h\hat{b}}+(\delta^b_f\delta^h_d-\delta^b_d\delta^h_f)D_{h\hat{a}}=0;$$
$$\delta^a_fD_{d\hat{b}}-\delta^a_dD_{f\hat{b}}+\delta^b_fD_{d\hat{a}}-\delta^b_dD_{f\hat{a}}=0;$$
$$(bf):\qquad D_{d\hat{a}}-\delta^a_d tr(D_{d\hat{a}})+3D_{d\hat{a}}-D_{d\hat{a}}=0;$$
$$D_{d\hat{a}}=\frac{1}{3}\delta^a_d tr(D_{d\hat{a}}).$$

2. Conversely, let

$$D_{d\hat{a}}=\frac{1}{3}\delta^a_d tr(D_{d\hat{a}}).$$

Then

$$B^{abc}+B^{bac}=\frac{1}{3\sqrt{2}}tr(D_{d\hat{a}})(\varepsilon^{ahb}\delta^c_h-\varepsilon^{ahc}\delta^b_h)$$
$$+\frac{1}{3\sqrt{2}}tr(D_{d\hat{a}})(\varepsilon^{bha}\delta^c_h-\varepsilon^{bhc}\delta^a_h)=$$

$= \frac{1}{3\sqrt{2}} tr(D_{d\hat{a}})\,(\varepsilon^{abc} - \varepsilon^{abc}) = 0$.

That is why $B^{abc} = -B^{bac}$.

So, THEOREM 1 is completely proved..

Corollary 1. The matrix (D_{kj}) determinig the configuration tensor of a six-dimensional nearly-Kählerian submanifolds of Cayley algebra has the following form:

$$(D_{kj}) = \left(\begin{array}{c|c} O & \bar{\mu} I_3 \\ \hline \mu I_3 & O \end{array}\right).$$

where I_3 is the identity matrix.

Knowing matrix (D_{kj}), we can exhaustively correct the Cartan structural equations (8) of a nearly-Kählerian $M^6 \subset \mathbf{O}$.

THEOREM 2. The Cartan structural equations of a six-dimensional nearly-Kählerian submanifold of Cayley algebra are the following:

$$d\omega^a = \omega^a_b \wedge \omega^b + \mu \varepsilon^{acb} \omega_b \wedge \omega_c;$$
$$d\omega_a = -\omega^b_a \wedge \omega_b + \bar{\mu} \varepsilon_{acb}\, \omega^b \wedge \omega^c; \qquad (9)$$
$$d\omega^a_b = \omega^a_c \wedge \omega^c_b + i\lambda T^7_{b[d}\, \delta^a_{c]} \omega^c \wedge \omega^d +$$
$$+((-\frac{1}{2}\delta^a_b \delta^c_d + \frac{3}{2}\delta^a_d \delta^c_b)|\lambda|^2 - 2T^7_{\hat{a}\hat{c}} T^7_{db})\omega_c \wedge \omega^d - i\bar{\lambda}\, T^7_{\hat{a}[\hat{d}} \delta^{\hat{b}}_{\hat{c}]}\, \omega_c \wedge \omega^d.$$

Taking into account that an almost Hermitian submanifold of Cayley algebra is Kählerian precisely when its matrix (D_{kj}) vanishes, we write down the Cartan structural equations of Kählerian $M^6 \subset \mathbf{O}$.

Corollary 2. The Cartan structural equations of a six-dimensional nearly-Kählerian submanifold of Cayley algebra are the following:

$$d\omega^a = \omega^a_b \wedge \omega^b\ ;$$
$$d\omega_a = -\omega^b_a \wedge \omega_b; \qquad (10)$$
$$d\omega^a_b = \omega^a_c \wedge \omega^c_b - 2T^7_{\hat{a}\hat{h}} T^7_{bg}\, \omega_h \wedge \omega^g.$$

We remark that this result was obtained by V.F. Kirichenko in a different way [29].

These structural equations contain all information about the geometry of such six-dimensional submanifolds of Cayley algebra. For instance, we can compute the spectra of some classical tensors of Kählerian $M^6 \subset \mathbf{O}$. In view of the reality and the symmetry properties of the tensor of Riemannian curvature [35], we conclude that only four types of components of this tensor determine its spectrum, namely:

R_{abcd}, $R_{\hat{a}bcd}$, $R_{\hat{a}\hat{b}cd}$, $R_{\hat{a}b\hat{c}d}$.

By direct computing we obtain the following result:

$R_{abcd} = 0$; $R_{\hat{a}bcd} = 0$; $R_{\hat{a}\hat{b}cd} = 0$;

$R_{\hat{a}b\hat{c}d} = -2T^7_{\hat{a}\hat{c}} T^7_{bd}$.

Now, we use the definition of the Ricci tensor [35]:

$ric_{kj} = R^m{}_{kjm}$.

That is why we get:

$ric_{ab} = R^c{}_{abc} + R^{\hat{c}}{}_{ab\hat{c}} = R_{cab\hat{c}} + R_{\hat{c}abc} = 0$;

$ric_{\hat{a}b} = R^c{}_{\hat{a}bc} + R^{\hat{c}}{}_{\hat{a}b\hat{c}} = R_{\hat{c}\hat{a}bc} + R_{c\hat{a}b\hat{c}} = R_{c\hat{a}b\hat{c}} = R_{\hat{a}c\hat{c}b} = -2T^7_{\hat{a}\hat{c}} T^7_{cb}$.

$ric_{a\hat{b}} = R^c{}_{a\hat{b}c} + R^{\hat{c}}{}_{a\hat{b}\hat{c}} = R_{\hat{c}a\hat{b}c} + R_{ca\hat{b}\hat{c}} = R_{\hat{c}a\hat{b}c} = -2T^7_{ac} T^7_{\hat{c}\hat{b}}$.

$ric_{\hat{a}\hat{b}} = R^c{}_{\hat{a}\hat{b}c} + R^{\hat{c}}{}_{\hat{a}\hat{b}\hat{c}} = R_{c\hat{a}\hat{b}\hat{c}} + R_{\hat{c}\hat{a}\hat{b}c} = 0$.

So, the tensor Ricci spectrum is the following:

$ric_{ab} = 0$;

$ric_{\hat{a}b} = -2T^7_{\hat{a}\hat{c}} T^7_{cb}$.

Next, we compute the spectrum of Weyl tensor of conformal curvature [35] defined by

$$W_{ijkl} = R_{ijkl} + \frac{1}{n-2}(ric_{ik}\, g_{jl} + ric_{jl}\, g_{ik} - ric_{il}\, g_{jk} - ric_{jk}\, g_{il}) +$$
$$+\frac{K}{(n-1)(n-2)}(g_{jk}\, g_{il} - g_{jl}\, g_{ik}),$$

where K is the scalar curvature.

Putting $n = 6$, by direct computing we obtain:

$W_{abcd} = 0$;

$W_{\hat{a}bcd} = 0$;

$$W_{\hat{a}\hat{b}cd} = -\frac{1}{2}\left(T^7_{\hat{a}\hat{h}} T^7_{hc}\, \delta^b_d + T^7_{\hat{b}\hat{h}} T^7_{hd}\, \delta^a_c - T^7_{\hat{a}\hat{h}} T^7_{hd}\, \delta^b_c - T^7_{\hat{b}\hat{h}} T^7_{hc}\, \delta^a_d\right) +$$
$$\frac{K}{20}\delta^{ba}_{cd}\ ;$$

$$W_{\hat{a}b\hat{c}d} = -2T^7_{\hat{a}\hat{c}} T^7_{bd} + \frac{1}{2}\left(T^7_{\hat{a}\hat{h}} T^7_{hd}\, \delta^c_b + T^7_{\hat{c}\hat{h}} T^7_{hb}\, \delta^a_d\right) + \frac{K}{20}\delta^c_b\, \delta^a_d\ .$$

Let us put together the obtained results. The spectra of the structural tensors and of the fundamental form are found from (1).

Table of classical tensors of six-dimensional Kählerian submanifolds of Cayley algebra

TENSOR	TENSOR SPECTRUM
Almost complex structure	$J^a_b = i\delta^a_b$, $J^{\hat{a}}_b = 0$, $J^a_{\hat{b}} = 0$, $J^{\hat{a}}_{\hat{b}} = -i\delta^b_a$
Riemannian metric	$g_{ab} = 0$, $g_{\hat{a}b} = \delta^a_b$, $g_{a\hat{b}} = \delta^b_a$, $g_{\hat{a}\hat{b}} = 0$
Fundamental form	$F_{ab} = 0$, $F_{\hat{a}b} = -i\delta^a_b$, $g_{a\hat{b}} = i\delta^b_a$, $g_{\hat{a}\hat{b}} = 0$
Riemannian curvature tensor	$R_{abcd} = 0$, $R_{\hat{a}bcd} = 0$, $R_{\hat{a}\hat{b}cd} = 0$, $R_{\hat{a}b\hat{c}d} = -2T^7_{\hat{a}\hat{c}} T^7_{bd}$
Ricci tensor	$ric_{ab} = 0$, $ric_{\hat{a}b} = -2T^7_{\hat{a}\hat{c}} T^7_{cb}$
Weyl tensor	$W_{abcd} = 0$, $W_{\hat{a}bcd} = 0$, $W_{\hat{a}\hat{b}cd} = -\frac{1}{2}\left(T^7_{\hat{a}\hat{h}} T^7_{hc}\, \delta^b_d + T^7_{\hat{b}\hat{h}} T^7_{hd}\, \delta^a_c - T^7_{\hat{a}\hat{h}} T^7_{hd}\, \delta^b_c - T^7_{\hat{b}\hat{h}} T^7_{hc}\, \delta^a_d\right) + \frac{K}{20}\delta^{ba}_{cd}$, $W_{\hat{a}b\hat{c}d} = -2T^7_{\hat{a}\hat{c}} T^7_{bd} + \frac{1}{2}\left(T^7_{\hat{a}\hat{h}} T^7_{hd}\, \delta^c_b + T^7_{\hat{c}\hat{h}} T^7_{hb}\, \delta^a_d\right) + \frac{K}{20}\delta^c_b\, \delta^a_d$.

We remark that these data define more exactly the results obtained on six-dimensional Hermitian (i.e. integrable almost Hermitian) submanifolds of Cayley algebra [3], [6].

We remind that the notion of conharmonic curvature tensor was introduced by Y. Ishii [26]. In accordance with the definition,

$$Ch(X,Y,Z,W)=R(X,Y,Z,W)-$$
$$-\frac{1}{n-2}\begin{bmatrix}\langle X,W\rangle Ric(Y,Z)-\langle X,Z\rangle Ric(Y,W)\\ +\langle Y,Z\rangle Ric(X,W)-\langle Y,W\rangle Ric(X,Z)\end{bmatrix}$$

Using the properties of the conharmonic curvature tensor of nearly-Kählerian manifolds established in recent works in this direction [36], [39], we deduce some results in the theory of six-dimensional nearly-Kählerian and Kählerian submanifolds of Cayley algebra.

THEOREM 4. A nearly-Kählerian six-dimensional submanifold of Cayley algebra is a conharmonically parakählerian if and only if it is a Ricci-flat Kählerian manifold.

THEOREM 5. A nearly-Kählerian six-dimensional submanifold of Cayley algebra is conharmonically flat if and only if it is holomorphically isometric to the complex Euclidean space C^3 with a canonical Kählerian structure.

THEOREM 6. A simply connected nearly-Kählerian six-dimensional submanifold of Cayley algebra is a manifold of pointwise constant holomorphic conharmonic curvature if and only if it is a manifold of global constant holomorphic conharmonic curvature.

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في أطياف بعض التنزرات للمنطوي الجزئي لكهلر وكهلر التقريبي ذات سداسية الأبعاد لجبر كيلي

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الملخص

درسنا المنطويات الجزئية لكوهلر والكوهلر التقريبي ذات سداسية الأبعاد لجبر كيلي , مع حساب الجبر الفعال لطيف بعض التنزرات المعتادة في المنطويات الجزئية. والتي برهنت ان الكوهلر التقريبي ذات سداسية الأبعاد للمنطويات الجزئية لجبر كيلي .يكون مستوي كونهورمني إذا وفقط إذا لبنية كوهلر الشرعية. C^3 كانت متشاكلة ومتماثلة مع الفضاء الاقليدي المعقد