Certain Classes of Univalent Functions With Negative Coefficients Defined By General Linear Operator

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ABSTRACT

In this study, a subclass $S_{\mu,\beta}^{c,m}(\mu, \beta, \delta)$ of an univalent function with negative coefficients which is defined by anew general Linear operator $H_{\mu,\beta}^{c,m}(\mu, \beta, \delta)$ have been introduced. The sharp results for coefficients estimators, distortion and closure bounds, Hadamard product, and Neighborhood, and this paper deals with the utilizing of many of the results for classical hypergeometric function, where there can be generalized to m-hypergeometric functions. A subclasses of univalent functions are presented, and it has involving operator $H_{\mu,\beta}^{c,m}(\mu, \beta, \delta)$ which generalizes many well-known. Denote A the class of functions and we have other results have been studied.

A function $f \in A$ is said to be starlike of complex order if the following condition (see[4]) is satisfied:

$$\text{Re}\left\{\frac{zf(z)^{-1}}{f(z)^{-1} - \mu\left(\frac{f(z)}{z}\right)}\right\} > \beta, \quad 0 \leq \mu < \frac{1}{2 \beta}, 0 < \beta \leq 1, 0 < \delta \leq 1$$

(1.2)

For complex parameters $c_1, \ldots, c_l$ and $b_1, \ldots, b_s$ (by $\ell \in \mathbb{C}\{0,1,2,\ldots\}$, $j=1,\ldots,\lfloor m \rfloor < 1$), the m-hypergeometric

$$\Psi_e = \sum_{n=0}^{\infty} \frac{\left(c_1,m\right)_n \ldots \left(c_l,m\right)_n}{\left(b_1,m\right)_n \ldots \left(b_s,m\right)_n} z^n$$

(1.3)

($t = r + 1$ such that $t, r \in \mathbb{N}_0 = \{0,1,2,3,4,\ldots\}$; $Z \in \mathbb{H}$).

The m-shifted factorial is involving by

$$(c,m)_n = 1 \quad \text{and} \quad (c,m)_n = (1-c)(1-cm)(1-cm^2) \ldots (1-cm^{n-1}), n \in \mathbb{N}$$

where $c$ any complex number and in terms of the Gamma function

$$\Gamma_{\mu,\beta}^{c,m}(\mu, \beta, \delta) = \frac{\Gamma_{\mu,\beta}^{(1+m)(1-cm^n)}}{\Gamma_{\mu,\beta}^{(1-m)}}$$

such that

$$\Gamma_{\mu,\beta}^{(1+m)(1-cm^n)} \Gamma_{\mu,\beta}^{(1-m)} = 0, \quad m < 1.$$

The study suggests that note that and by utilizing ratio test, the series (1.3) converges absolutely in open unit disk $U$, $|z|<1$.

$\Psi_e = \sum_{n=0}^{\infty} \frac{\left(c_1,m\right)_n \left(c_2,m\right)_n \ldots \left(c_l,m\right)_n}{\left(b_1,m\right)_n \ldots \left(b_s,m\right)_n} z^n$ ($m < 1, z \in \mathbb{U}$)

Is the m-Gauss hypergeometric function see [4],[5].
Recently Mohammed and Darus [1] defined the following:
\[ I(c; b; m) f : A \rightarrow A \]
\[ I(c; b; m) f \equiv z + \sum_{n=2}^{\infty} \frac{(c nm_{n-1} - (c m_{n-1}) a_n z^n}{n+c} . \]
The Srivastava-Attiya operator \( T_{s,c} \) is defined in [6] as:
\[ T_{s,c} f(z) = z + \sum_{n=2}^{\infty} \frac{(1+c) a_n z^n}{n+c} , \quad (1.4) \]
where \( z \in U \), \( c \in \mathbb{C}/\{0, -1, -2, \ldots \} \), \( s \in \mathbb{C} \) and \( f \in A \).
This linear operator \( T_{s,c} \) can be written as:
\[ T_{s,c} f(z) = g_{s,c} (z) * f(z) = (1+c) (z, s, c) \]
by utilizing the Hadamard product (convolution). Here,
\[ \Phi (z, s, c) = \sum_{n=0}^{\infty} \frac{z^n}{(n+c)^s} , \]
is the well-known Hurwitz-Lerch zeta function (see[6],[7]). It is also an important function of Analytic Number Theory such as the De-Jonquiere function:
\[ H(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+c)} , \quad (Re(s) > 1) \{ |z| = 1 \} . \]
We can define the class \( \mathcal{H}_m^{s,c} (c_i; b_i)(f) \) as follows:
\[ \mathcal{H}_m^{s,c} (c_i; b_i)(f) \equiv z + \sum_{n=2}^{\infty} \frac{(c_{nm} - (c_{nm-1}) a_n z^n}{n+c} . \]
A class \( \mathcal{R}_m^{s,c} (\mu, \beta, \delta) \) is defined in [3] by:
\[ \mathcal{R}_m^{s,c} (\mu, \beta, \delta) = \mathcal{R}_m^{s,c} (\mu, \beta, \delta) \in \Gamma \]
and the study have following class and confirms that note that by specializing the parameters \( \mu, \beta, \delta \)
1. The class \( \mathcal{S}_m^{s,k_0} (a, \beta, \delta) \) is the class studied by A. R.S.Juma and M. Darus [8].
2. The class \( \mathcal{S}_m^{s,k_0} (0, 1, 1) \) is precisely the class of starlike function in \( U \).
3. The class \( \mathcal{S}_m^{s,k_0} (\mu, 1, 1) \) is the class of starlike function of order \( \mu (0 \leq \mu < 1) \).
4. The class \( \mathcal{S}_m^{s,k_0} (0, \beta, s^{k_0}) \) is the class studied by Lakshmikantham-simhan[9].
5. The class \( \mathcal{S}_m^{s,k_0} (\mu, \beta, \delta) \) is the class studied by S. R. Kulkarni [10].

2. Coefficients estimates and Other properties

**Theorem 1.** Let \( f \) be defined by (1.7). Then \( f \in \mathcal{S}_m^{s,k_0} (\mu, \beta, \delta) \) if and only if
\[ \sum_{n=2}^{\infty} \frac{(c_{nm} - (c_{nm-1}) a_n z^n}{n+c} [(n - 1)(1 - \beta) - 2(1 - \mu)] a_n \leq 2(1 - \mu) \]
\[ 0 < \beta \leq 1, 0 \leq \mu < \frac{1}{2} \delta, \quad \frac{1}{2} \leq \delta \leq 1 \]

**Proof:** If \( |z|=1 \), then
\[ |z(\mathcal{H}_m^{s,c} (c_i; b_i)(f))| = |(\mathcal{H}_m^{s,c} (c_i; b_i)(f))| - |\mu(\mathcal{H}_m^{s,c} (c_i; b_i)(f))| \]
\[ -|\beta(\mathcal{H}_m^{s,c} (c_i; b_i)(f))| - |(\mathcal{H}_m^{s,c} (c_i; b_i)(f))| \]
By utilizing (1.5) we have
\[ \mathcal{H}_m^{s,c} (c_i; b_i)(f) \equiv z + \sum_{n=2}^{\infty} \frac{(c_{nm} - (c_{nm-1}) a_n z^n}{n+c} \]
\[ \beta(2(1 - \mu) + \sum_{n=2}^{\infty} \frac{(c_{nm} - (c_{nm-1}) a_n z^n}{n+c} \leq (1 + \beta) \]
\[ \left| \sum_{n=2}^{\infty} \frac{(c_{nm} - (c_{nm-1}) a_n z^n}{n+c} \leq \delta \right| \]

recently, Mohammed and Darus [1] defined the following:
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This linear operator \( T_{s,c} \) can be written as:
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5. The class \( \mathcal{S}_m^{s,k_0} (\mu, \beta, \delta) \) is the class studied by S. R. Kulkarni [10].
And versa, suppose that $\in S^c_m(\mu, \beta, \delta)$, therefore the condition (1.7) gives us
\[
\left| \frac{\sum_{n=2}^{\infty} \left( \frac{z}{n+1} \right) f(z)}{\sum_{n=2}^{\infty} \left( \frac{z}{n+1} \right) f(z)} \right| < \beta
\]
\[
\Rightarrow \left| \sum_{n=2}^{\infty} \left( \frac{z}{n+1} \right) f(z) \left( \frac{z}{n+1} \right) f(z) \right| \leq \beta
\]
\[
\Rightarrow \left| \sum_{n=2}^{\infty} \left( \frac{z}{n+1} \right) f(z) \left( \frac{z}{n+1} \right) f(z) \right| \leq \beta
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\Rightarrow \left| \sum_{n=2}^{\infty} \left( \frac{z}{n+1} \right) f(z) \left( \frac{z}{n+1} \right) f(z) \right| \leq \beta
\]

By theorem 2.1, we have
\[
\left| \sum_{n=2}^{\infty} \left( \frac{z}{n+1} \right) f(z) \left( \frac{z}{n+1} \right) f(z) \right| \leq \beta
\]

Then we have
\[
\left| \sum_{n=2}^{\infty} \left( \frac{z}{n+1} \right) f(z) \left( \frac{z}{n+1} \right) f(z) \right| \leq \beta
\]
\[
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\]

Then we have
\[
\left| \sum_{n=2}^{\infty} \left( \frac{z}{n+1} \right) f(z) \left( \frac{z}{n+1} \right) f(z) \right| \leq \beta
\]

Thus (2.3) is true. Further
\[
\left| \sum_{n=2}^{\infty} \left( \frac{z}{n+1} \right) f(z) \left( \frac{z}{n+1} \right) f(z) \right| \leq \beta
\]

And also
\[
\left| \sum_{n=2}^{\infty} \left( \frac{z}{n+1} \right) f(z) \left( \frac{z}{n+1} \right) f(z) \right| \leq \beta
\]

The result is sharp for function $f(z)$, defined by
\[
f(z) = z + \frac{2\beta}{\beta(1-\mu) + 1} z^2, z = \pm r.
\]

**Theorem 3.** Let $0 < \beta < 1$, $\mu_1 < \mu_2 < 1$ and $\delta_1 \leq 1$ the $S^c_m(\mu_2, \delta_2, \beta) \subset S^c_m(\mu_1, \delta_1, \beta)$.

**Proof:** By utilizing assumption we get
\[
\sum_{n=2}^{\infty} \left( \frac{z}{n+1} \right) f(z) \left( \frac{z}{n+1} \right) f(z) \leq \beta
\]

Therefore
\[
\sum_{n=2}^{\infty} \left( \frac{z}{n+1} \right) f(z) \left( \frac{z}{n+1} \right) f(z) \leq \beta
\]

The above bounds are sharp.

**Proof.** By theorem 1, we have
\[
\sum_{n=2}^{\infty} \left( \frac{z}{n+1} \right) f(z) \left( \frac{z}{n+1} \right) f(z) \leq \beta
\]

It is sufficient to show that $g(z) \in S^c_m(\mu, \beta, \delta)$ that mean
Definition 3.11: Let \( \gamma \geq 0 \), \( f(z) \in T \) on the \((k, \gamma)\)-neighbourhood of a function \( f(z) \) defined by

\[ N_\gamma f(z) = \{ g \in T : g(z) = z - \sum_{n=1}^{\infty} b_n z^n \} \text{ and } \sum_{n=2}^{\infty} |a_n - b_n| \leq \gamma. \]

For the identity function \( e(z) = z \), we get

\[ N_{\gamma} \gamma(z) = \{ g \in T : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \} \text{ and } \sum_{n=2}^{\infty} |b_n| \leq \gamma. \]
Then, the Hadamard product $h(z)$ defined by

$$h(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$$

is in the subclass $S_{m,c}^{\infty}(\mu,\beta,\delta)$ when

$$\mu_2 \leq \left\{ \begin{aligned}
(n-1)(1-\beta) + 2\beta(\mu - n - 1) \frac{\sum_{(c,m)_{m-1}}^{(c,m)_{n-1}} \frac{1} {((n-1)(1-\beta)+2\beta(\mu-n-1))}}{(1+c)_n + \beta(1+c)_n} \\
-2\beta(1-\mu_2)^2(n-1)(1-\beta) - (2\beta\delta)^2(1-\mu_2)^2/n + (1+c)^2 - (2\beta\delta)^2(1-\mu_2)^2
\end{aligned} \right\} \leq 1.$$ 

**Proof.** By theorem 1, we get

$$\sum_{n=2}^{\infty} (c_{m-1}^{(c,m)_{n-1}}(1+c)_n + \beta(1+c)_n) a_n b_n \leq \frac{2\beta(1-\mu_2)}{1-\beta}(1-\beta)^{1/2} \leq 1.$$ 

We get only to find the largest $\mu_2$ such that.

$$\sum_{n=2}^{\infty} (c_{m-1}^{(c,m)_{n-1}}(1+c)_n + \beta(1+c)_n) a_n b_n \leq 1.$$ 

Now by Cauchy–Schwarz inequality, we get

$$\sum_{n=2}^{\infty} (c_{m-1}^{(c,m)_{n-1}}(1+c)_n + \beta(1+c)_n) a_n b_n \leq \left[ \sum_{n=2}^{\infty} (c_{m-1}^{(c,m)_{n-1}}(1+c)_n + \beta(1+c)_n) a_n b_n \right]^{1/2} \leq 1.$$ 

Also, we need to prove that

$$\sum_{n=2}^{\infty} (c_{m-1}^{(c,m)_{n-1}}(1+c)_n + \beta(1+c)_n) a_n b_n \leq 1.$$ 

By theorem 1 and since $\frac{d}{d\alpha} \leq 1$ we get

$$\sum_{n=2}^{\infty} \frac{(c_{m-1}^{(c,m)_{n-1}}(1+c)_n + \beta(1+c)_n) a_n b_n}{2\beta(1-\mu_2)} \leq 1.$$ 

Then $h(z)$ is in $S_{m,c}^{\infty}(\mu,\beta,\delta).$

**Theorem 9.** Let $f(z) \in S_{m,c}^{\infty}(\mu,\beta,\delta)$ be defined by

$$f(z) = z + \frac{\sum_{n=2}^{\infty} (a_n n^\alpha - q_n^\alpha)}{z + \sum_{n=2}^{\infty} (a_n n^\alpha - q_n^\alpha)}.$$ 

Then $f(z)$ is also in $S_{m,c}^{\infty}(\mu,\beta,\delta).$

**Proof.** By virtue of theorem 1, $f(z)$ follows from (1.7) that

$$f(z) = \frac{\sum_{n=2}^{\infty} (a_n n^\alpha - q_n^\alpha)}{z + \sum_{n=2}^{\infty} (a_n n^\alpha - q_n^\alpha)},$$

But

$$\sum_{n=2}^{\infty} \frac{(c_{m-1}^{(c,m)_{n-1}}(1+c)_n + \beta(1+c)_n) a_n b_n}{2\beta(1-\mu_2)} \leq 1.$$ 

Since $\frac{d}{d\alpha} \leq 1$ and by theorem 1, so the proof is complete.

**Theorem 10.** Let $F(z) \in S_{m,c}^{\infty}(\mu,\beta,\delta)$ be defined by

$$F(z) = (1-\alpha)z + \gamma z^{\gamma} f(z) \left( \frac{1}{\omega} \right)$$

Then $F(z)$ is also in $S_{m,c}^{\infty}(\mu,\beta,\delta)$ if $0 \leq \alpha \leq 2.$

**Proof.** Let $f$ defined by (1.7). Then

$$F(z) = (1-\alpha)z + \gamma z^{\gamma} f(z) \left( \frac{1}{\omega} \right)$$

By theorem 1 and since $\frac{d}{d\alpha} \leq 1$ we get

$$\sum_{n=2}^{\infty} \frac{(c_{m-1}^{(c,m)_{n-1}}(1+c)_n + \beta(1+c)_n) a_n b_n}{2\beta(1-\mu_2)} \leq 1.$$ 

Then $f(z)$ is in $S_{m,c}^{\infty}(\mu,\beta,\delta).$

**References**


فئات معينة من دوال أحادية التكافؤ مع معاملات سالبة معرفة بواسطة العامل الخطي

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الملخص

في هذا البحث تم دراسة الفئات الفرعية من الدوال الأحادية التكافؤ مع معاملات سالبة والتي هي معرفة بواسطة العامل الخطي العام الذي قد قدم . وتم الحصول على النتائج في مقدرات المعاملات والتشوهات وضرب هادمرد ونتائج أخرى تم دراستها وكذلك في هذا البحث تم استخدام العديد من الدوال الكلاسيكية الهندسية العليا بحيث تستطيع أن تكون من نوع من نوع 

$H_m^{\alpha, \beta}(c, b)$

مع معاملات الخطي $S_m^{\alpha, \beta}(\mu, \beta, s)$ وتتضمن تدفق فئات فرعية .