Tikrit Journal of Pure Science Vol. 24 (6) 2019

TJPS

Tikrit Journal of Pure Science

ISSN: 1813 – 1662 (Print) --- E-ISSN: 2415 – 1726 (Online)

Journal Homepage: http://tjps.tu.edu.iq/index.php/j

Some properties on extended eigenvalues and extended eigenvectors

Laith K. Shaakir[1] , Anas A. Hijab[2]

[1]*Department of Mathematics, College of computer sciences and Mathematics, Tikrit University, Tikrit , Iraq*

[2]*Department of Mathematics, College of Education for Pure Sciences, Tikrit University , Tikrit , Iraq*

DOI: http://dx.doi.org/10.25130/tjps.24.2019.119

ARTICLE INFO.

Article history:

-Received: 21 / 1 / 2019

-Accepted: 3 / 7 / 2019

-Available online: / / 2019

Keywords: Extended eigenvalues, extended eigenvectors, unilateral shift operator, bilateral shift operator.

Corresponding Author:

Name:

E-mail: anas_abass@tu.edu.iq

Tel:

ABSTRACT

In this paper, the study extended eigenvalues and extended eigenvectors, and we will investigate the $E_\lambda(A)$ and give for some concepts properties and result important, also we will find the $E(U)$ and $E(B)$ on the ℓ^2 space, so U is Unilateral shift operator and $B = U^*$.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space, and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. A complex number λ is called an extended eigenvalue of an operators $A \in \mathcal{B}(\mathcal{H})$ if there exists a non-zero operator $X \in \mathcal{B}(\mathcal{H})$ satisfying the equation $AX = \lambda XA$, such an operator X is called extended eigenvector for the operator A corresponding to λ. The eigenvalue terminology, although not perfectly accurate, seems useful on two levels. The first was described in [1], briefly, if A has dense range, then the equation

$AX = \varphi(X)A, \varphi(X) \in \mathcal{B}(\mathcal{H})$ .... (1)

has a unital algebra as its solution set, and φ is a unital homomorphism. Our extended eigenvalues are precisely the eigenvalues for φ. The second point of view is that one can easily show that for an operator on a finite dimensional space, the set of extended eigenvalues for that operator is the set of ratios of eigenvalues, with the obvious restriction on the use of 0. This is shown explicitly in [2]. More see [3, 4]. Now the set of all extended eigenvalues of A is denoted by $E(A)$ and the set of all extended eigenvectors of A corresponding to λ is denoted by $\tilde{E}_\lambda(A)$ and $E_1(A)$ is$\{A\}'$, the commutant of A, that is, the set of all operators commuting with A. In this paper, we give some concepts properties and we prove important theorems. In particular we find $E(U)$ also $E(B)$ on the ℓ^2 space.

2. Extended eigenvalues and Extended eigenvectors

Definition (2.1) :[2] A complex number λ is called an extended eigenvalue of $A \in \mathcal{B}(\mathcal{H})$ if there exists a non-zero operator $X \in \mathcal{B}(\mathcal{H})$ satisfying the equation

$AX = \lambda XA$ .... (2)

Such that an operator X is called extended eigenvector corresponding to λ, is denoted by $E(A)$ to the set of all extended eigenvalues for A; that is, $E(A) = \{\lambda \in \mathbb{C}: \exists 0 \neq X \in \mathcal{B}(\mathcal{H}),\ AX = \lambda XA\}$ and $\tilde{E}_\lambda(A)$ is the set of all extended eigenvectors corresponding to λ , therefore $\tilde{E}_\lambda(A) = \{0 \neq X \in \mathcal{B}(\mathcal{H}),\ AX = \lambda XA\}$. $E(A)$ is non-empty set, since $1 \in E(A)$ and the identity operator I $\in \tilde{E}_1(A)$.

Proposition (2.2) : Suppose that $A \in \mathcal{B}(\mathcal{H})$. Then $E_\lambda(A) = \tilde{E}_\lambda(A) \cup \{0\}$ is closed linear subspace of $\mathcal{B}(\mathcal{H})$.

Proof: Let $X, Y \in E_\lambda(A)$, and $\alpha,\ \beta, \lambda \in \mathbb{C}$. So $AX = \lambda XA$, and $AY = \lambda YA$.

Thus $A(\alpha X + \beta Y) = \alpha AX + \beta AY = \alpha\lambda XA + \beta\lambda YA = \lambda(\alpha X + \beta Y)$. Therefore $(\alpha X + \beta Y) \in E_\lambda(A)$.

Then $E_\lambda(A)$ is subspace of $\mathcal{B}(\mathcal{H})$. Let $\{X_n\}$ be a sequence of operators in $E_\lambda(A)$ converges to X. So

that $AX_n = \lambda X_n A$ for every positive integer n. Since A is continues and $\{X_n\} \longrightarrow X$, implies that $\{AX_n\} \longrightarrow AX$, and $\{\lambda X_n A\} \longrightarrow \lambda XA$. Thus $AX = \lambda XA$. Then $X \in E_\lambda(A)$. It is clear that, if A is the zero operator, then $E(A) = \mathbb{C}$ and $\tilde{E}_\lambda(A) = \mathcal{B}(\mathcal{H})\backslash\{0\}$, for each $\lambda \in E(A)$. ■

Lemma (2.3) : If A is a non-zero nilpotent operator, then $E(A) = \mathbb{C}$.

Proof: Since A is nilpotent operator, then there exist a positive integer n, such that $A^n = 0$ and $A^{n-1} \neq 0$. If λ is any complex number, then $\lambda A^{n-1}A = \lambda A^n = 0$ and $AA^{n-1} = A^n = 0$. So that $AA^{n-1} = \lambda A^{n-1}A$. Since $A^{n-1} \neq 0$. Then $\lambda \in E(A)$ or $A^{n-1} \in \tilde{E}_\lambda(A)$. ■

Recall that the wiener algebra $W(\mathbb{D})$ is the set of all analytic function f on the unit disc $\mathbb{D}$ such that $W(\mathbb{D}) := \{f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in Hol(\mathbb{D}) : \|f\| = \sum_{n=0}^{\infty} |\hat{f}(n)| < +\infty\}$. [5].

The unilateral shift operator S on $W(\mathbb{D})$ is defined by $Sf(z) = zf(z)$, $f \in W(\mathbb{D})$. In the following theorem, M.Gurdal fined the extended eigenvalue for S.

Theorem (2.4) :[5] If S is the Unilateral shift operator on $W(\mathbb{D})$, then $E(S) = \{\lambda : |\lambda| \geq 1\}$.

Recall that the unilateral shift operator U on the space $\ell^2 = \{x = (x_1, x_2, x_3, \ldots) : \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$ is defined by $U(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$ for every $(x_1, x_2, x_3, \ldots) \in \ell^2$. In the following theorem, we prove that, theorem above is true when the unilateral shift U is defined on ℓ^2 the space.

Theorem (2.5) : If U is the Unilateral shift operator on ℓ^2, then $E(U) = \{\lambda : |\lambda| \geq 1\}$.

Proof: Suppose that $|\lambda| \geq 1$ and $D_{1/\lambda}$ is the diagonal operator on ℓ^2. So that $D_{1/\lambda}(x_1, x_2, x_3, \ldots) = (x_1, {}^1/_\lambda x_2, {}^1/_{\lambda^2} x_3, \ldots)$ for every $(x_1, x_2, x_3, \ldots) \in \ell^2$. It's clear that $D_{1/\lambda}$ is non-zero bounded operator and $UD_{1/\lambda} = \lambda D_{1/\lambda} U$. Therefore $\lambda \in E(U)$ and $D_{1/\lambda}$ is extended eigenvector for λ.

Now, we shall prove that if $|\lambda| < 1$, then λ not extended eigenvalue. It is clear $\lambda \in E(U)$ with that $\lambda = 0$ is not extended eigenvalue, since U is injective. So that we assume $0 < |\lambda| < 1$. Therefore there exists non-zero operator X satisfy $UX = \lambda XU$. Let $\{e_n\}$ be the standard basis for ℓ^2 (i.e.) $e_n = (0, 0, \ldots, \underbrace{1}_{nth}, 0, \ldots)$, $n = 1, 2, \ldots$. Now $\|Xe_{n+1}\| = \|XUe_n\| = {}^1/_{|\lambda|} \|UXe_n\| = {}^1/_{|\lambda|} \|Xe_n\|$ Therefore $\|Xe_{n+1}\| = {}^1/_{|\lambda|^{n+1}} \|Xe_1\| \longrightarrow \infty$, as $n \longrightarrow \infty$, which this is contradict that X is bounded operator. Thus $E(U) = \{\lambda : |\lambda| \geq 1\}$.■

In the following corollary, we find the extended eigenvalues for the bilateral shift operator B such that $B(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$ for every $(x_1, x_2, x_3, \ldots) \in \ell^2$, another word $B^* = U$.

Corollary (2.6) : If B is bilateral shift operator on ℓ^2, then $E(B) = \{\lambda : |\lambda| \leq 1\}$.

Proof: Let $|\lambda| \leq 1$ and D_λ is the diagonal operator, so that $BD_\lambda(x_1, x_2, x_3, \ldots) = B(x_1, \lambda x_2, \lambda^2 x_3, \ldots) = (\lambda x_2, \lambda^2 x_3, \lambda^3 x_4, \ldots) = \lambda D_\lambda(x_2, x_3, x_4, \ldots) = (\lambda x_2, \lambda^2 x_3, \lambda^3 x_4, \ldots)$ for every $(x_1, x_2, x_3, \ldots) \in \ell^2$. Thus $\lambda \in E(B)$. Then $\{\lambda : |\lambda| \leq 1\} \subseteq E(B)$ and D_λ is extended eigenvector.

Now, assume $\lambda \in E(B)$ with $|\lambda| > 1$, then there exists a non-zero operator X such that $BX = \lambda XB$, by taking adjoint, we have $B^*X^* = {}^1/_{\bar{\lambda}} X^*B^*$. Since the $B^* = U$, we have ${}^1/_{\bar{\lambda}} \in E(U)$, this is contradict the fact $E(U) = \{\lambda : |\lambda| \geq 1\}$. Thus $E(B) \subseteq \{\lambda : |\lambda| \leq 1\}$. So $E(B) = \{\lambda : |\lambda| \leq 1\}$. ■

Proposition (2.7) : Let $T, S, C \in \mathcal{B}(\mathcal{H})$. Then:

(1) If $S \in E_\alpha(T)$, then $T \in E_{1/\alpha}(S)$, where $\alpha \neq 0$.

(2) $\tilde{E}_\alpha(S)^* = \tilde{E}_{1/\bar{\alpha}}(S^*)$. Where $\tilde{E}_\alpha(S)^* = \{X : X^* \in \tilde{E}_\alpha(S)\}$.

(3) If $T \in E_\alpha(S)$ and $S \in E_\beta(C)$ where $\beta \neq 0$, then $TC \cap CT \in E_{\alpha/\beta}(S)$. In particular if $S \in \{C\}'$, then $TC \cap CT \in E_\alpha(S)$.

(4) If $T \in E_\alpha(S) \cap E_\beta(C)$, then $T \in E_{\alpha\beta}(SC) \cap E_{\alpha\beta}(CS)$. In particular, when $T \in \{C\}'$, then $T \in E_\alpha(SC) \cap E_\alpha(CS)$.

Proof: (1) Since $S \in E_\alpha(T)$, then $TS = \alpha ST$. Hence $ST = {}^1/_\alpha TS$, implies that $T \in E_{1/\alpha}(S)$.

(2) Let $X \in \tilde{E}_\alpha(S)^*$, then $X^* \in \tilde{E}_\alpha(S)$. Hence $X^* \neq 0$ and $SX^* = \alpha X^*S$. So that $S^*X = {}^1/_{\bar{\alpha}} XS^*$. Thus $X \in \tilde{E}_{1/\bar{\alpha}}(S^*)$, then $\tilde{E}_\alpha(S)^* \subseteq \tilde{E}_{1/\bar{\alpha}}(S^*)$. By the same way we can show that $\tilde{E}_{1/\bar{\alpha}}(S^*) \subseteq \tilde{E}_\alpha(S)^*$. Thus $\tilde{E}_\alpha(S)^* = \tilde{E}_{1/\bar{\alpha}}(S^*)$.

(3) Let $T \in E_\alpha(S)$, $S \in E_\beta(C)$ and $\beta \neq 0$, then $ST = \alpha TS$ and $CS = \beta SC$. So $CST = \alpha CTS$. Therefore $S(CT) = {}^\alpha/_\beta (CT)S$, then $CT \in E_{\alpha/\beta}(S)$ also by same way, implies that $TC \in E_{\alpha/\beta}(S)$. If $\beta = 1$. Hence $TC \cap CT \in E_\alpha(S)$.

(4) Since $T \in E_\alpha(S)$ and $T \in E_\beta(C)$. Then $ST = \alpha TS$ and $CT = \beta TC$. Thus $CST = \alpha CTS$. So $(CS)T = \alpha\beta T(CS)$, implies that $T \in E_{\alpha\beta}(CS)$ also $STC = \alpha TSC$. Thus $(SC)T = \alpha\beta(SC)T$. Therefore $T \in E_{\alpha\beta}(SC)$. ■

Recall that if that A and B are two bounded operators on Hilbert spaces $\mathcal{H}$ then A is similar to B if there exists invertible operator $T \in \mathcal{B}(\mathcal{H})$ such that $AT = TB$, we denote by $A \sim B$, when A is similar to B. If $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$, then A is quasi-similar to B if there exists operators T_1 from $\mathcal{H}_1$ to $\mathcal{H}_2$ and T_2 from $\mathcal{H}_2$ to $\mathcal{H}_1$ such that both T_1 and T_2 are injective with dense ranges such that $T_1A = BT_1$ and $AT_2 = T_2B$, we denote by $A \approx B$, when A is quasi-similar to B.

Proposition (2.8) :[2] Suppose that operators A and B are quasi-similar. Then $E(A) = E(B)$.

Corollary (2.9) : Suppose that operators A and B are similar and C is quasi-similar to A or B. Then $E(A) = E(B) = E(C)$.

Lemma (2.10): Suppose that λ is extended eigenvalue for the operator A, such that $\lambda^n = 1$ for some positive integer n. Then $E_\lambda(A)^{n+1} \subset E_\lambda(A)$.

Proof: Suppose that $X \in E_\lambda(A)$. So that satisfy formula (1). $AX = \lambda XA$. Thus $AX^{n+1} = \lambda^{n+1}X^{n+1}A$ since $\lambda^n = 1$ for some $n \in N$. So $AX^{n+1} = \lambda X^{n+1}A$. Hence $X^{n+1} \in E_\lambda(A)$. Then $E_\lambda(A)^{n+1} \subset E_\lambda(A)$. ∎

Lemma (2.11) : Suppose that A is bounded operator on a Hilbert space $\mathcal{H}$, such that $0 \in E(A)$, then:

(1) A is not injective.

(2) A is not invertible.

(3) A^* does not have dense range.

(4) A is not unitary.

Proof: Suppose that $0 \in E(A)$. Then there exists operator $T \neq 0$ satisfying, $AT = 0$. So that A is not injective. For the proof of (2) follows from (1).

(3) Suppose that A^* has dense range, since $\ker(A) = Rang(A^*)^\perp$. Then $\ker(A) = \{0\}$. Hence A is injective, this contradict (1).

(4) If A is unitary operator, then $A^*A = AA^* = I$. This contradict (2). ∎

We see in theorem (2.5) that the extended eigenvalues of the unilateral shift operator is $E(U) = \{\lambda : |\lambda| \geq 1\}$. However the U is not invertible. Therefore the converse of lemma (2.10) is not true.

3. Conclusion

This paper has presented a deserving sets of operators called extended eigenvalues and extended eigenvectors. Some of the characters of unilateral and bilateral operators were studied and fined. The described work is focused on relationship between concepts and properties of extended eigenvalues and extended eigenvectors. As for future work is concerned, generalized of set for two operators.

References

[1] Biswas, A., Lambert, A., and Petrovic, S. (2002). Extended eigenvalues and the Volterra operator. *Glasg. Math. J*, **44(3)**; 521-534.

[2] Biswas, A. and Petrovic, S. (2006). On extended eigenvalues of operators. *Integral Equations and Operator Theory*, **55(2)**; 233-248.

[3] Laruz, M.; Saavedra, F.; Petrovic, S. and Zabeti, O. (2014). Extended eigenvalues for Cesàro operators. *arXiv: Math. FA*, **1:**1403-4844.

[4] Mansour, A. and Hechifa, A. (2016). Some Interesting properties of finite continuous Cesàro operators. *AJMAA*, **13(1)**; 1-6.

[5] Gurdal M. (2009). On the extended eigenvalues and extended eigenvectors of shift operator on the Wiener algebra. *Applied. Math. Letters*, **22**:1727-1729.

[6] Lambert, A. (2004). Hyperinvariant subspaces and extended eigenvalues. *New York. J. Math*, **10**:83-88.

بعض الخواص حول القيم الذاتية الموسعة والمتجهات الذاتية الموسعة

ليث خليل شاكر ، أنس عباس حجاب

[1] قسم الرياضيات، كلية علوم الحاسوب والرياضيات، جامعة تكريت ، تكريت ، العراق

[2] قسم الرياضيات، كلية التربية للعلوم الصرفة، جامعة تكريت ، تكريت ، العراق

الملخص

تهدف الدراسة الحالية لتبيان القيم الذاتية الموسعة والمتجهات الذاتية الموسعة. كما بينت الدراسة بعض المفاهيم والخواص للمجموعة $E_\lambda(A)$ مع بعض النتائج المهمة، وكذلك وجدنا مجموعة القيم الذاتية لمؤثر النقلة $E(U)$ و $E(B)$ على الفضاء ℓ^2.