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α -almost similar operators

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ABSTRACT

The study focuses on α -almost similar operator which is a new concept of the operator theory and also some basic concepts related to the concept α -almost similar.

The study also defines a new concept called β -operator which is an expansion of the concept θ -operator and the relationship of this concept with the α -almost similar.

At the end of this research, we study some important relationships among similar, unitarily equivalent, and almost similar on the one hand and α -almost similar on the other.

Introduction

We denote $B(\mathcal{H}_1, \mathcal{H}_2)$ to the set of all bounded linear operators from a Hilbert space \mathcal{H}_1 into a Hilbert space \mathcal{H}_2 . if $\mathcal{H} = \mathcal{H}_1 = \mathcal{H}_2$ then we denote $B(\mathcal{H})$ instead of $B(\mathcal{H}_1, \mathcal{H}_2)$. The operator $T \in B(\mathcal{H})$ is called self-adjoint if $T = T^*$ where T^* is the adjoint of T [1]. An operator $A \in B(\mathcal{H})$ is said to be isometric if $A^*A = I$ [2]. If $A^*A = AA^*$ then A is called normal operator. And if $A^*A = AA^* = I$ then A is said to be unitary [3]. If $A^* = A$ and $A^2 = A$ then A is said to be projection. If $AA^*A = A$ then A is said to be partially isometric, equivalently A^*A is projection (i.e. $(A^*A)^2 = A^*A$) [4]. Clearly every unitary operator is isometric and normal.

Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{H})$ are said to be similar and denoted by $A \sim B$, if there exists an invertible operator X such that $XA = B$ (equivalently $A = X^{-1}BX$). If $A \sim B$, then A and B have the same: spectrum, point spectrum and approximate point spectrum [5].

Similarly, two operators $A, B \in B(\mathcal{H})$ are said to be unitarily equivalent and denoted by $A \cong B$, if there exists a unitary operator U such that $UA = BU$ (equivalently $A = U^*BU$) [4]. If A, B are similar normal then they are unitarily equivalent by Fuglede-Putnam theorem [6].

Let A, B be two bounded linear operators on $B(\mathcal{H})$. Then A, B are said to be almost similar and denoted

by $A \stackrel{a.s.}{\sim} B$ if there exists an invertible operator X such that:

$A^*A = X^{-1}B^*BX$ and, $A^* + A = X^{-1}(B^* + B)X$. The class of almost similar was first introduced by Jibril [7]. we have extended this concept to α -almost similar and demonstrated some different results.

An operator $A \in B(\mathcal{H})$ is said to be θ -operator if A^*A commutes with $A^* + A$. The class of all θ -operator in $B(\mathcal{H})$ is denoted by θ . The class of θ -operator which has been widely studied by Campbell [8]. We have extended the concept of θ -operator to another concept we called it β -operator, the class of β -operator in $B(\mathcal{H})$ is denoted by β .

Let $T \in B(\mathcal{H})$ then the set of all complex number λ for which $T - \lambda I$ is not invertible is called the spectrum of T and denoted by $\sigma(T)$ that is, $\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is not invertible}\}$. The complement of the spectrum of T is called resolvent set of T . The spectrum of T can be split into many disjoint sets [9]. The point spectrum of the operator T is denoted by $\sigma_p(T)$ is the set of all those λ for which $T - \lambda I$ is not injective, that is $\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(T - \lambda I) \neq \{0\}\}$.

A scalar λ is said to be the approximate point spectrum for the operator T and denoted by $\sigma_{ap}(T)$, if there exists a sequence of unit vector $\{x_n\}$ such that $\|(T - \lambda I)x_n\| \rightarrow 0$ [9]. Let T be a linear

transformation from a normed space X into a normed space Y (i.e. $T: X \rightarrow Y$). Then T is said to be compact if $\overline{T(\mathcal{B})}$ is compact for every bounded subset \mathcal{B} of X , that is, $\overline{T(\mathcal{B})}$ is relatively compact for every bounded subset \mathcal{B} of X [9].

1. Basic concept on α -almost similarity

Definition 1.1: Let α be a real number, two bounded linear operators $A, B \in B(\mathcal{H})$ are said to be α -almost similar and, denoted by $A \approx B$. If there exist an invertible operator X such that:

$$A^*A = X^{-1}B^*BX \quad \dots\dots\dots(1) \quad \text{and} \quad A^* + \alpha A = X^{-1}(B^* + \alpha B)X \quad \dots\dots\dots(2).$$

Example 1.2: Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ be the operators on the two-dimensional Hilbert space \mathbb{C}^2 , and define the invertible operator on \mathbb{C}^2 as follows:

$X = X^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, take $\alpha = 2$, then $A \approx B$. To show that

$$A^*A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X^{-1}B^*BX \quad ($$

$$\begin{aligned} & \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X^{-1}B^*BX \\ & A^* + 2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \\ & 2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ & = X^{-1}(B^* + 2B)X \end{aligned}$$

Remark 1.3: Every 1- almost similar operators are almost similar and the converse are true.

The following example show almost similar and α -almost similar are independent when $\alpha \neq 1$.

Example 1.4: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ be the operators on the two-dimensional Hilbert space \mathbb{C}^2 , and define the invertible operator on \mathbb{C}^2 as follows: $X = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$, take $\alpha = -1$. Then $A \approx B$. But

A is not almost similar to B . Since $A^* + A \neq X^{-1}(B^* + B)X$, indeed $A^* + A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I$, $B^* + B = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$. $B^* + B \neq X(A^* + A)X^{-1} = 2XIX^{-1} = 2I$ for every invertible operator X .

Theorem 1.5: let $\alpha \in \mathbb{R}$, the relation \approx on $B(\mathcal{H})$ is equivalence relation.

Proof: (i) Reflexivity, let $A \in B(\mathcal{H})$ take $X = I$. $A^*A = X^{-1}A^*AX$ and, $A^* + \alpha A = X^{-1}(A^* + \alpha A)X$. Then $A \approx A$.

(ii) Symmetry, suppose that $A, B \in B(\mathcal{H})$ and, $A \approx B$. Then there exists an invertible operator X such that.

$$A^*A = X^{-1}B^*BX \quad \dots\dots\dots(1), \quad \text{and} \quad A^* + \alpha A = X^{-1}(B^* + \alpha B)X \quad \dots\dots\dots(2).$$

Now, pre-multiplying and post-multiplying (1) and (2) by X and X^{-1} , respectively yields. $XA^*AX^{-1} = B^*B \quad \dots\dots\dots(3)$, and, $X(A^* + \alpha A)X^{-1} = B^* + \alpha B \quad \dots\dots\dots(4)$.

Take $Y = X^{-1}$, which is an invertible operator, since X^{-1} is an invertible operator.

Substituting X and X^{-1} in (3) and (4) by Y^{-1} and Y respectively, we get $B \approx A$.

(iii) Transitivity, suppose that A, B and $C \in B(\mathcal{H})$. And $A \approx B$, $B \approx C$, to show that $A \approx C$.

Since $A \approx B$, then there exists an invertible operator X such that.

$$A^*A = X^{-1}B^*BX \quad \dots\dots\dots(1), \quad \text{and} \quad A^* + \alpha A = X^{-1}(B^* + \alpha B)X \quad \dots\dots\dots(2).$$

Also, since $B \approx C$, then there exists an invertible operator $Y \in B(\mathcal{H})$ such that

$$B^*B = Y^{-1}C^*CY \quad \dots\dots\dots(3) \quad \text{and} \quad B^* + \alpha B = Y^{-1}(C^* + \alpha C)Y \quad \dots\dots\dots(4).$$

Substituting (3) and (4) in (1) and (2) as follows:

$$A^*A = X^{-1}[Y^{-1}C^*CY]X = X^{-1}Y^{-1}[C^*C]YX = (YX)^{-1}C^*C(YX) \quad \dots\dots\dots(5)$$

Also, $A^* + \alpha A = X^{-1}[Y^{-1}(C^* + \alpha C)Y]X$. Which implies that $A^* + \alpha A = (YX)^{-1}[C^* + \alpha C](YX) \quad \dots\dots\dots(6)$. Then from (5) and (6) we get $A \approx C$.

Proposition 1.6: Let $A \in B(\mathcal{H})$, such that $A \approx 0$, then $A = 0$.

Proof: Since $A \approx 0$ then there exists an invertible operator X such that.

$$A^*A = X^{-1}0^*0X = 0 \quad \dots\dots\dots(1), \quad \text{and} \quad A^* + \alpha A = X^{-1}(0^* + \alpha 0)X = 0 \quad \dots\dots\dots(2).$$

Then $A^*A = 0$ and $A^* + \alpha A = 0$. Now, $\|Ax\|^2 = \langle Ax | Ax \rangle = \langle A^*Ax | x \rangle = \langle 0 | x \rangle = 0$

Therefore $Ax = 0$ for all $x \in \mathcal{H}$. Thus $A = 0$.

Remark 1.7: suppose that $A, B \in B(\mathcal{H})$ such that $A \approx B$, then clearly by using mathematical induction we can prove:

- (i) $(A^*A)^n = X^{-1}(B^*B)^nX$,
- (ii) $(A^* + \alpha A)^n = X^{-1}(B^* + \alpha B)^nX$. For all-natural number n .

Proposition 1.8: Let $A, B \in B(\mathcal{H})$ such that $A \approx B$. Then A is isometric if and only if B is isometric.

Proof: Suppose that A is isometric. Since $A \approx B$ this means that there exists an invertible operator X such that $A^*A = X^{-1}(B^*B)X \quad \dots\dots\dots(1)$, and, $A^* + \alpha A = X^{-1}(B^* + \alpha B)X \quad \dots\dots\dots(2)$. Since A is isometric then $A^*A = I$ substituting in the equality (1) we have $I = A^*A = X^{-1}(B^*B)X$ which implies that $B^*B = I$. Thus, B is isometric.

Conversely: by the same way we can prove that A is isometric whenever B is isometric.

Proposition 1.9: Let $\alpha \in \mathbb{R}$, A, B are two operators in $B(\mathcal{H})$ with $A \approx B$. Then:

- (i) A^*A is onto if and only if B^*B is onto,
- (ii) $A^* + \alpha A$ is onto if and only if $B^* + \alpha B$ is onto,
- (iii) A^*A is one-to-one if and only if B^*B is one-to-one,
- (iv) $A^* + \alpha A$ is one-to-one if and only if $B^* + \alpha B$ is one-to-one,
- (v) A^*A is projection if and only if B^*B is projection.

Proof: Clearly.

Remark 1.10: Let $\alpha \in \mathbb{R}$, A, B are two operators in $B(\mathcal{H})$ with $A \approx B$. Then:

- (vi) A^*A is one-to-one and onto if and only if B^*B is one-to-one and, onto.
- (vii) $A^* + \alpha A$ is one-to-one and, onto if and only if $B^* + \alpha B$ is one-to-one and, onto.

Proof: immediately from proposition 1.9 above.

proposition 1.11: Let $A \in B(\mathcal{H})$ and $A \approx I$, then A is isometry.

Proof: Suppose that $A \approx I$. then there exists an invertible operator X such that $A^*A = X^{-1}(I^*I)X = X^{-1}(I)X = X^{-1}X = I \dots (1)$. Then $A^*A = I$ (i.e. A is isometry).

Proposition 1.12: Let $A, B \in B(\mathcal{H})$ and $A \approx B$ such that A is partially isometric then B is partially isometric.

Proof: $A \approx B$ means that there exists an invertible operator X such that

$A^*A = X^{-1}(B^*B)X \dots (1)$. Since A is partially isometric then A^*A is projection (i.e. $(A^*A)^2 = A^*A$). By squaring both sides in (1) we have $(X^{-1}(B^*B)X)(X^{-1}(B^*B)X) = (A^*A)^2 = A^*A$. Then $X^{-1}(B^*B)(B^*B)X = X^{-1}(B^*B)X \dots (2)$.

Pre-multiplying and post-multiplying (2) by X and X^{-1} respectively we have, $(B^*B)^2 = B^*B$ (i.e. B^*B is projection). Which implies that B is partially isometric.

Proposition 1.13: Let $\alpha \in \mathbb{R}$. Then the transformation $\varphi : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ that satisfies $\varphi(A^*A) = X^{-1}(B^*B)X$, $\varphi(A^* + \alpha A) = X^{-1}(B^* + \alpha B)X$ is an automorphism. That is, it maps sums into sums, products into products and scalar multiples into scalar multiples.

Proof: suppose that A, B, C and $D \in B(\mathcal{H})$ such that $\varphi(A^*A) = X^{-1}(B^*B)X$ and $\varphi(C^*C) = X^{-1}(D^*D)X$. Then

$$\begin{aligned} \varphi(A^*A + \alpha C^*C) &= X^{-1}(B^*B + \alpha D^*D)X = \\ &= X^{-1}(B^*B)X + \alpha X^{-1}(D^*D)X = \varphi(A^*A) + \alpha \varphi(C^*C) \text{ and, } \varphi((A^*A)(C^*C)) = X^{-1}((B^*B)(D^*D))X \\ &= X^{-1}(B^*B)XX^{-1}(D^*D)X \\ &= (X^{-1}(B^*B)X)(X^{-1}(D^*D)X) = \varphi(A^*A)\varphi(C^*C). \end{aligned}$$

Proposition 1.14: Let $A, B \in B(\mathcal{H})$ such that A, B are unitarily equivalent then $A \approx B$ for every $\alpha \in \mathbb{R}$.

Proof: Since A and B are unitarily equivalent then there exists a unitary operator U such that $A = U^*BU$. Then $A^* = U^*B^*U$ which implies that $A^*A = (U^*B^*U)(U^*BU) = U^*B^*(UU^*)BU = U^*B^*(I)BU = U^*B^*BU$. And, $A^* + \alpha A = U^*B^*U + \alpha U^*BU = U^*B^*U + U^*\alpha BU = U^*(B^* + \alpha B)U$

Thus, $A \approx B$ for all $\alpha \in \mathbb{R}$.

proposition 1.15: Let $A, B \in B(\mathcal{H})$ such that $A \approx B$ for every real α . Then $(A + \lambda I) \approx (B + \lambda I)$ for every real λ .

Proof: $A \approx B$ means that there is an invertible operator X such that.

$$A^*A = X^{-1}(B^*B)X \dots (1). \text{ And, } A^* + \alpha A = X^{-1}(B^* + \alpha B)X \dots (2).$$

From the equality (2) we have $A^* + \alpha A = X^{-1}B^*X + X^{-1}\alpha B X$, by post-adding to both sides $\lambda I + \alpha \lambda I$ which implies that $A^* + \alpha A + \lambda I + \alpha \lambda I = X^{-1}B^*X + X^{-1}\alpha B X + \lambda I + \alpha \lambda I$. Then we have $A^* + \lambda I + \alpha(A + \lambda I) = X^{-1}B^*X + X^{-1}\alpha B X + \lambda I + \alpha \lambda I$ which implies that

$$\begin{aligned} (A + \lambda I)^* + \alpha(A + \lambda I) &= \\ X^{-1}(B + \lambda I)^*X + X^{-1}(\alpha B + \lambda I)X \dots (3). \end{aligned}$$

Since λ is real number. Now, we want to prove that $(A +$

$$\begin{aligned} \lambda I)^*(A + \lambda I) &= X^{-1}(B + \lambda I)^*(A + \lambda I)X. \quad (A + \lambda I)^*(A + \lambda I) = A^*A + \lambda A^* + \lambda A + \lambda^2 I = A^*A + \lambda(A^* + A) + \lambda^2 I \\ &= X^{-1}(B^*B)X + \lambda X^{-1}(B^* + B)X + \lambda^2 X^{-1}X \text{ (since (1) and (2) are satisfied when } \alpha = 1) = X^{-1}[(B^*B) + \lambda(B^* + B) + \lambda^2 I]X = X^{-1}[(B^* + \lambda I)(B + \lambda I)]X \\ &= X^{-1}[(B + \lambda I)^*(B + \lambda I)]X, \text{ since } \lambda \text{ is real number.} \\ \text{Then } (A + \lambda I)^*(A + \lambda I) &= X^{-1}[(B + \lambda I)^*(B + \lambda I)]X \dots (4). \end{aligned}$$

From the equality (3) and the equality (4) we have $(A + \lambda I) \approx (B + \lambda I)$ for every real λ .

proposition 1.16: Let $A, B \in B(\mathcal{H})$ be projections such that $A \approx B$ and $(A + \lambda I) \approx (B + \lambda I)$. Then: $\sigma(A) = \sigma(B)$, $\sigma_p(A) = \sigma_p(B)$ and $\sigma_{ap}(A) = \sigma_{ap}(B)$.

Proof: $A \approx B$ means that there is an invertible operator X such that.

$$A^*A = X^{-1}(B^*B)X \dots (1). \text{ And, } A^* + \alpha A = X^{-1}(B^* + \alpha B)X \dots (2).$$

Since A and B are projection then A and B are self-adjoints. Then (2) becomes $(1 + \alpha)A = X^{-1}(1 + \alpha)B X$ which implies that $A = X^{-1}B X$. This means that $A \sim B$, then

$$\sigma(A) = \sigma(B), \sigma_p(A) = \sigma_p(B) \text{ and, } \sigma_{ap}(A) = \sigma_{ap}(B) [6].$$

Theorem 1.17 [10]: the operator $A \in B(\mathcal{H})$ is compact if and only if A^*A is compact.

Proposition 1.18: Let $\alpha \in \mathbb{R}$, $A, B \in B(\mathcal{H})$ and $A \approx B$. If A is compact then B is compact.

Proof: since $A \approx B$ then there exist an invertible operator X such that

$A^*A = X^{-1}B^*B X$ pre-multiplying and post-multiplying both sides by X and X^{-1} respectively, we have $X A^*A X^{-1} = B^*B$. Since A is compact then $X A^*A X^{-1}$ is also compact. By theorem 1.17 above then B is compact.

Theorem 1.19: Let $\alpha \in \mathbb{R}$, $A, B \in B(\mathcal{H})$, X be an invertible operator. If $XA = BX$ and, $XA^* = B^*X$. Then A and B are α -almost similar.

Proof: by hypothesis $XA = BX$ and, $XA^* = B^*X$ then we have $A = X^{-1}BX$ and, $A^* = X^{-1}B^*X$. Now, $A^*A = (X^{-1}B^*X)(X^{-1}BX) = X^{-1}B^*(X X^{-1})BX = X^{-1}B^*B X$ and,

$$A^* + \alpha A = X^{-1}B^*X + X^{-1}(\alpha B)X = X^{-1}(B^* + \alpha B)X. \text{ Then } A \text{ and } B \text{ are } \alpha\text{-almost similar.}$$

Proposition 1.20: If $A, B \in B(\mathcal{H})$ are similar normal operators, then $A \approx B$.

Proof: suppose that A and B are similar normal operators then there exists an invertible operator X such that $XA = BX$. Then $XA^* = B^*X$ by Fuglede-Putnam theorem [6].

Now, by using theorem 1.20 we have, A and B are α -almost similar.

Remark 1.21: The converse of the proposition 1.20 is not true in general.

Consider the following example: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $X = X^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, be the operators on

two-dimensional Hilbert space \mathbb{C}^2 , take $\alpha = 2$, then $A \approx B$. Also A is similar to B (I.e. $XA = BX$) but $A^*A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = AA^*$ and, $B^*B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = BB^*$. Then A and B are not normal operators.

2. The properties of self-adjoint operator on α -almost similarity.

Proposition 2.1: Suppose that A, B are self-adjoint operators in $B(\mathcal{H})$ with $A \sim B$ (i.e. A is similar to B), then $A \approx B$, for every $\alpha \in \mathbb{R}$.

Proof: Since A and B are similar operators, then there exists an invertible operator X such that $XA = BX$ (i.e. $A = X^{-1}BX$).

Also, A and B are self-adjoint operators in $B(\mathcal{H})$, then

$A^*A = X^{-1}B^*BX$ (1). Also, $A^* + \alpha A = A + \alpha A = X^{-1}BX + \alpha X^{-1}BX = X^{-1}(B + \alpha B)X = X^{-1}(B^* + \alpha B)X$ (2). From (1) and (2) we have $A \approx B$.

Remark 2.2: The converse of the Proposition 2.1. above is not true in general.

For example: Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $X = X^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, be the operators on the two-dimensional Hilbert space \mathbb{C}^2 take $\alpha = 2$. We know that $A \approx B$ as in example 1.2. Moreover $A \sim B$. But $A \neq A^*$, also $B \neq B^*$. Thus, A and B are not self-adjoint operators.

Proposition 2.3: Let $\alpha = -1 \in \mathbb{R}$. $A, B \in B(\mathcal{H})$ and $A \approx B$. If A is self-adjoint then B is self-adjoint.

Proof: Since $A \approx^{-1} B$, then there exist an invertible operator X such that $A^* - A = X^{-1}(B^* - B)X$. Which implies that $0 = X^{-1}(B^* - B)X$ (1). Pre-multiplying and post multiplying (1) by X and X^{-1} respectively we have $0 = B^* - B$. Then $B = B^*$.

Remark 2.4: The converse of proposition 2.3 above is not true in general for example $A = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} =$

$A^*, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = B^*$ and, $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the operators on the two-dimensional Hilbert space \mathbb{C}^2 , take $\alpha \in \mathbb{R}$

Then $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 0 \end{bmatrix} \neq X^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = X^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = X^{-1}I X = I$. Thus, A is not α -almost similar to B . Then A is not (-1) -almost similar to B .

Theorem 2.5 [4]: (Cartesian form) let T be any operator, then there exist self-adjoint operators A and B such that $T = A + iB$. When $A = \frac{1}{2}(T + T^*)$ and, $B = \frac{1}{2i}(T - T^*)$.

Theorem 2.6: Let $T \in B(\mathcal{H})$ then $T = T^*$ if and only if T is normal and

$$(T + T^*)^2 = 4T^*T.$$

Proof: If $T = T^*$ then clearly $(T + T^*)^2 = 4T^*T$ and T is normal.

Conversely: If $4T^*T = (T^* + T)^2 = (T^* + T)(T^* + T) = T^{*2} + 2T^*T + T^2$. Hence, $T^{*2} - 2T^*T + T^2 =$

0 which implies that $(T^* - T)^2 = 0 \Rightarrow -(T^* - T)^2 = 0 \Rightarrow (T^* - T)(T - T^*) = 0$. Let $S = T^* - T \Rightarrow SS^* = 0 \Rightarrow 0 = \langle SS^*x | x \rangle = \langle S^*x | S^*x \rangle = \|S^*x\|^2$ for every x . Then $S^*x = 0$ for every $x \Rightarrow S^* = 0 \Rightarrow S = 0 \Rightarrow T^* - T = 0 \Rightarrow T^* = T$.

Remark 2.6: If $T = T^*$ then $(T^* + \alpha T)^2 = (1 + \alpha)^2 T^*T$ for every $\alpha \in \mathbb{R}$.

Proposition 2.7: Suppose that $(T^* + \alpha T)^2 = (1 + \alpha)^2 T^*T$ then:

(i) If $\alpha = 1$ and T is normal then $T = T^*$.

(ii) If $\alpha = -1$ then $T = T^*$.

(iii) If $\alpha \neq 1, -1$ then $T^{*2} = T^2$.

Proof: (i) directly as in theorem 2.6. And (ii) clearly.

Now to prove (iii) let $\alpha \neq 1, -1$. $(T^* + \alpha T)^2 = (1 + \alpha)^2 T^*T$ by taking adjoint to both sides we have $(T + \alpha T^*)^2 = (1 + \alpha)^2 T^*T$. Then $T^{*2} + \alpha T^*T + \alpha T T^* + \alpha^2 T^2 = T^2 + \alpha T T^* + \alpha^2 T^{*2} \Rightarrow T^{*2} = T^2$.

Theorem 2.8 [4]: If T is normal operator, then there exists a unitary operator U such that $T^* = UT$.

3. The properties of β -operator on α -almost similarity.

Definition 3.1: let $A \in B(\mathcal{H})$, then A is called an β -operator if A^*A commutes with $A^* + \alpha A$. The class of all β -operator in a Banach algebra on a Hilbert space \mathcal{H} is denoted by β i.e. $\beta = \{A : A \in B(\mathcal{H}) \text{ such that } [A^*A, A^* + \alpha A] = 0\}$.

Example 3.2: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and take $\alpha = 3$ then $[A^*A, A^* + 3A] = 0$

i.e. $(A^*A)(A^* + 3A) = (A^* + 3A)(A^*A)$ which implies that A is β -operator.

Proposition 3.3: If $A \in B(\mathcal{H})$ is β -operator then kA is β -operator for every real number k .

Proof: Clearly.

Proposition 3.4: If $A, B \in B(\mathcal{H})$ and $A \approx B$ such that B is β -operator then A is β -operator.

Proof: $A \approx B$ means that there exists an invertible operator X such that

$$A^*A = X^{-1}(B^*B)X. \text{ And, } A^* + \alpha A = X^{-1}(B^* + \alpha B)X.$$

Then, $[X^{-1}(B^* + \alpha B)X][X^{-1}(B^*B)X] = [A^* + \alpha A]A^*A$ (1)

And $[X^{-1}(B^*B)X][X^{-1}(B^* + \alpha B)X] = A^*A[A^* + \alpha A]$ (2). From the equality (1) we have: $[X^{-1}(B^* + \alpha B)(B^*B)X] = [A^* + \alpha A]A^*A$ (3).

Also, from the equality (2) we have: $[X^{-1}(B^*B)(B^* + \alpha B)X] = A^*A[A^* + \alpha A]$ (4).

Since B is β -operator then the left-hand side of the equality (3) and the equality (4) are equal. which imply that the right-hand side of the equality (3) and the equality (4) are equal. Hence A is β -operator.

4. The relation among similarity, unitarily equivalent, quasi similarity and almost similarity with α -almost similarity.

Proposition 4.1: Let $A, B \in B(\mathcal{H})$ are orthogonal projection then A and B are α -almost similar if and only if A and B are similar.

Proof: Suppose that $A \approx B$ and A, B are projection then by proposition 1.16 we get $A \sim B$.

Conversely, suppose that A and B are similar operators then there exists invertible operator X such that $A = X^{-1}B X$, since A and B are orthogonal projection then $A = A^* = A^2$, $B = B^* = B^2$. Which implies that $A^2 = X^{-1}B^2 X$ then we have $A^*A = X^{-1}B^*B X$.

On the other hand, the second inequality follows from the fact that

$$A^* + \alpha A = (1 + \alpha)A = (1 + \alpha)X^{-1}B X = X^{-1}(B^* + \alpha B)X. \text{ Thus, } A \approx B.$$

Proposition 4.2: Let $\alpha \in \mathbb{R}$. $A, B \in B(\mathcal{H})$ and A, B are self-adjoint then A and

B are unitarily equivalent if and only if $A \approx B$.

Proof: Suppose that A and B are unitarily equivalence then by proposition 1.14 we have $A \approx B$.

Conversely: Suppose that $A, B \in B(\mathcal{H})$ are self-adjoint with $A \approx B$.

Now, $A \approx B$ means that there exists an invertible operator X such that

$$A^*A = X^{-1}(B^*B) X \dots (1), \text{ and } A^* + \alpha A = X^{-1}(B^* + \alpha B) X \dots (2).$$

Since A, B are self-adjoint and $A \approx B$ then they are similar operators (i.e. $A = X^{-1}B X$). Then A and B are both similar and self-adjoint operators then A and B are normal. Thus A and B are unitarily equivalence.

Corollary 4.3: Let $\alpha \in \mathbb{R}$. $A, B \in B(\mathcal{H})$ are self-adjoint and $A \approx B$. Then A and B are unitarily equivalent.

Proof: directly from proposition 4.2 above.

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Proposition 4.4: Let $A, B \in B(\mathcal{H})$ are self-adjoint operators then, A and B are α -almost similar if and only if A and B are almost similar.

Proof: Suppose that A, B are α -almost similar then there is an invertible operator X such that. $A^*A = X^{-1}(B^*B) X \dots (1)$, and $A^* + \alpha A = X^{-1}(B^* + \alpha B) X \dots (2)$.

Since A and B are self-adjoint Then $A = A^*$, $B = B^*$ then (2) becomes

$$(1 + \alpha)A = (1 + \alpha)X^{-1}B X. \text{ Now pre-multiplying both sides by } \frac{2}{(1+\alpha)}, \alpha \neq -1. \text{ Which implies that } 2A = 2X^{-1}BX \Rightarrow A + A^* = X^{-1}(B + B^*)X \dots (3).$$

From (1) and (3) we have A and B are almost similar.

Conversely, suppose that A, B are almost similar then (1) and (3) satisfies. Since A and B are self-adjoint Then (3) becomes $2A = 2X^{-1}BX$. pre-multiplying both sides by $\frac{1+\alpha}{2}$ which implies that $(1 + \alpha)A = (1 + \alpha)X^{-1}BX \Rightarrow A + \alpha A = X^{-1}(B + \alpha B) X \Rightarrow A^* + \alpha A = X^{-1}(B^* + \alpha B) X$. Thus, A and B are α -almost similar.

Remark 4.7: the converse of proposition 4.6 is not true in general consider the following example: Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ be the operators on the two-dimensional Hilbert space \mathbb{C}^2 , and define the invertible operator on \mathbb{C}^2 as follows: $X = X^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, take $\alpha = 2$. then $A \approx B$. As in example 1.2. Also, $A \approx^s B$. But $A \neq A^*$ and, $B \neq B^*$.

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المؤثرات الخطية المتشابهة تقريبا من النمط- α

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الملخص

درسنا في هذه البحث المؤثرات الخطية المقيدة المتشابهة تقريبا من النمط- α وهو مفهوم جديد لنظرية المؤثرات الخطية، كذلك بعض المفاهيم الاساسية المتعلقة بمفهوم المؤثرات الخطية المقيدة المتشابهة تقريبا من النمط- α . كذلك عرفنا مفهومين جديدا والذي اطلقنا عليه اسم المؤثر من النمط- β والذي يعتبر توسيعا للمؤثر من النمط- θ وعلاقة هذا المؤثر بالمؤثرات الخطية المتشابهة تقريبا من النمط- α . في نهاية هذا البحث درسنا بعض العلاقات المهمة بين لتشابه، والمؤثرات الاحادية المتكافئة، والتشابه التقريبي من جهة وبين المؤثرات الخطية المتشابهة تقريبا من النمط- α من الجهة الاخرى.