The Orthogonality of Martingale in Birkhoff’s sense and others
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ABSTRACT
Orthogonality is one of an important concepts in Mathematics, therefore it will be discussed in this paper, the orthogonality of martingale according to Birkhoff’s, Roberts’s, Singer’s, Carlsson’s sense for orthogonality and the conditions that are needed to have orthogonality.

1. Introduction
Let \( \{X_n, n \geq 1\} \) be a sequence of integrable random variable on a probability space \((\Omega, \mathcal{F}, P)\) and 
\( \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \) an increasing sequence of sub field of \( \mathcal{F} \), \( X_n \) is \( \mathcal{F}_n \)- measurable that is 
\( X_n : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \). \( \{X_n, \mathcal{F}_n\} \) is said to be martingale iff
\[ \forall n, E[X_n | \mathcal{F}_n] = X_{n-1} \ a.e. \] [1]
If 
\[ E(\| Y \|^2) < \infty \] It’s said \( Y \in L^2(\Omega, F, P) \) for
\( P \in [1, \infty) \) and the norm defined as :
\[ \| Y \|_2 = \left\{ E(\| Y \|^2) \right\}^{1/2}. \] [2]
James was the first one Who studied Birkhoff’s properties of orthogonality, therefor this orthogonality is called Birkhoff-James.[3]
Ash proved that the martingale difference is orthogonal in a Hilbert space \( L^2(\Omega, F, P) \) by usual orthogonality . [1]
In 1934 Roberts introduced his orthogonality as Roberts’s orthogonality and in 1935 Birkhoff introduced his orthogonality as Birkhoff’s orthogonality, which was one of the most important orthogonality senses in normed space . [4]
Singer’s orthogonality was introduced by singer in 1957 .Therefore, the orthogonality of martingale will be discussed in normed space according to these senses.

2- Main results
Definition(2.1)[7]
In a vector normed space \( L^\infty(\Omega, F, P) \) , \( Z \) is called Birkhoff – James orthogonal to \( W \) and denoted by \( Z \perp_B W \) if
\[ \| Z + aW \|_\infty \geq \| Z \|_\infty \]
for any real number \( a \).
Theorem (2.2)[7]
Let $X$ and $Y$ belong to a normed space $L^p(\Omega,F,P)$, then $X \perp Y$ if and only if there exist $g \in L^p \setminus \{0\}$ such that $|g(X)| = \|g\|_p \|X\|_p$.

Lemma (2.3)
Let $E : L^p(\Omega,F,P) \rightarrow \overline{B}$ be a conditional expectation, then $E$ is bounded linear operator.

Proof:
Since $E(aX + bY \mid F) = aE(X \mid F) + bE(Y \mid F)$ then $E$ is linear operator and
$$\|E(X \mid F)\|_p \leq \|X\|_p$$
that is $E$ is bounded by $K = 1$.

Theorem (2.4)
Let $X_\alpha \in L^p(\Omega,F,P)$ and $(X_\alpha, F_\alpha)$ be a zero – mean martingale then $X_\alpha \perp X_{\alpha+1}$.

Proof: 
If $g(X_\alpha) = E(X_\alpha \mid F_\alpha)$, we define $|g(X_\alpha)| = \|g(X_\alpha)\|_p$.
$$|g(X_\alpha)| = \|E(X_\alpha \mid F_\alpha)\|_p$$
$$\|X_\alpha\|_{E(1|F_\alpha)}$$ (since $X_\alpha$ is $F_\alpha$- measurable)
$$\|E(1|F_\alpha)\|_p \|X_\alpha\|_p$$ (by property of norm)
$$\|E\|_p \|X_\alpha\|_p$$
$$\|g\|_p \|X_\alpha\|_p$$.

Since $|g(X_\alpha)| = \|g\|_p \|X_\alpha\|_p$ and $E(X_{\alpha+1} \mid F_\alpha) = 0$, then (by (2.2))
$$\sum_{n=1}^{\infty} \|\beta X + \gamma Y\|_p^2 = 0$$
$X_\alpha \perp X_{\alpha+1}$.

Definition (2.5)[8]
In a normed space $L^p(\Omega,F,P)$, $Z$ is said to be singular orthogonal to $W$ and denoted by $Z \perp W$ if either $\|Z\|_p \|W\|_p = 0$ or $\|Z + W\|_p = \|Z - W\|_p$.

Theorem (2.6)
If $E(|X_n|^p) < \infty$ for all $n$, $X_n$ is a martingale and independent if $E(X_{\alpha+1}^p \mid F_\alpha) = 0$, $\forall P \in [1, \infty)$ then $X_\alpha \perp X_{\alpha+1}$.

Proof: 
$$\|X_n\|_p \|X_{\alpha+1}\|_p = E(|X_n|^p)E(|X_{\alpha+1}|^p)$$
$$E[ X_n^p X_{\alpha+1}^p ]$$ (by independence)
$$E[E(X_n^p X_{\alpha+1}^p \mid F_n)]$$
$$E[X_n^p X_{\alpha+1}^p]$$ (Since $X_n^p$ is $F_n$ - measurable).

Therefore $\|X_n\|_p \|X_{\alpha+1}\|_p = 0$
Hence $X_\alpha \perp X_{\alpha+1}$.

Definition (2.7)[4]

In a normed space $L^p(\Omega,F,P)$, $Z$ is said to be Roberts orthogonal to $W$ and denoted by $Z \perp_{\alpha} W$ if the equality
$$\|Z + \alpha W\|_p = \|Z - \alpha W\|_p$$ holds for any real number $\alpha$.

Theorem (2.8)
If $X_n$ belong to a Hlibert space $L^2(\Omega,F,P)$ and $(X_n, F_n)$ is a martingale then martingale differences are orthogonal in the sense of Roberts.

Proof: 
$$\|X_n - X_{n-1}\|_2 \|F_n\| = X_n - X_{n-1} = 0$$ (by property of martingale)
$$E(X_n^2 - X_{n-1}) + \alpha^2 E(X_n - X_{n-1})^2$$
$$\|X_n - X_{n-1}\|^2 + \alpha^2 \|X_n - X_{n-1}\|^2$$ ... (1)

Similarity
$$\|X_n - X_{n-1}\|^2 - \alpha^2 \|X_n - X_{n-1}\|^2$$
$$\|X_n - X_{n-1}\|^2 + \alpha^2 \|X_n - X_{n-1}\|^2$$ ... (2)

From (1) and (2) we have,
$$\|X_n - X_{n-1}\|^2 + \alpha^2 \|X_n - X_{n-1}\|^2$$
$X_n \perp X_{n-1}$.

Definition (2.9)
Let $X$, $Y$ belong to a normed space $L^p(\Omega,F,P)$, and $m$ be a positive integer. Then $X$ is said to be orthogonal in the sense of Carlson to $Y$ and denoted by $X \perp_{\alpha} Y$ if and only if
$$\sum_{i=1}^{\infty} \alpha_i \|\beta_i X + \gamma_i Y\|_p = 0$$

Where $\alpha_i, \beta_i, \gamma_i$ are real number such that
$$\sum_{i=1}^{\infty} \alpha_i \beta_i^2 = \sum_{i=1}^{\infty} \alpha_i \gamma_i^2 = 0, \quad \sum_{i=1}^{\infty} \alpha_i \beta_i \gamma_i = 1$$

Theorem (2.10)
Let $E(|X_n|^p) < \infty$ and $X_n$ be a martingale such that $E(X_n^p \mid F_n) = 0$, $\forall P \in [1,\infty)$ and $K = 0,1,\ldots$ then $X_{n+1} \perp_{\alpha} X_n$.

Proof: 
$$\sum_{i=1}^{\infty} \alpha_i \|\beta_i X_{n+1} + \gamma_i Y_i\|_p^2 = \sum_{i=1}^{\infty} \alpha_i \left\{ (E(\beta_i X_{n+1} + \gamma_i Y_i)^2) \right\}^p$$

(by binomial theorem)
\[
\alpha_k = \sum_{i=1}^{n} \left( \sum_{j=0}^{p-1} \left( \sum_{i=1}^{n} \beta_{i} \right) \gamma_{i} E \left[ X_{n}^{p-k} X_{n}^{k} \right] \right)^{2} \]

\[
= 0
\]

Since \( X_n^K \) is \( F_n \) - measurable.

Hence \( X_{n+1} \perp_{c} X_n \).

References


