Tikrit Journal of Pure Science
ISSN: 1813 – 1662 (Print) --- E-ISSN: 2415 – 1726 (Online)
Journal Homepage: http://tjps.tu.edu.iq/index.php/j

A Simulation Study of Some Restricted Estimators in Restricted Linear Regression Model

Bader Aboud Mohammad , Mustafa Ismaeel Naif
Department of Mathematics, College of Education For Pure Science, University Of Anbar, Anbar, Iraq
DOI: http://dx.doi.org/10.25130/tjps.26.2021.053

ARTICLE INFO.
Article history:
-Received: 1 / 3 / 2021
-Accepted: 15 / 4 / 2021
-Available online: / / 2021
Keywords: Multicollinearity, Linear regression model, Restricted biased estimators, Mean square error, Simulation study.
Corresponding Author:
Name: Bader Aboud Mohammad
E-mail:
bad19u2003@uoanbar.edu.iq
Tel:

ABSTRACT

When the multicollinearity exists in linear regression model, the result of the Restricted Least Square estimator (RLS) is unstable. So that, more researchers proposed the restricted biased estimators to improve the efficiency of RLS estimator. In this paper, Some of biased restricted estimators have been introduced to study the performance of them. The simulation study has been given to compare these estimators. According to simulation study, we found that, the shrinkage restricted ridge regression (SRRE) estimator which proposed by Baber and Mustafa [1], has good properties comparing with other restricted estimators that given in this study. A Numerical example has been considered to illustrate the performance of these estimators

1. Introduction

Consider the standard linear regression model

$Y = X\beta + e,$ (1)

where Y is an $n \times 1$ vector of the response variable, X is an $n \times p$ matrix of the explanatory variables, β is a $1 \times p$ vector of the unknown parameters and e is an $n \times 1$ vector of the random errors with the mean $E(e) = 0$ and the variance $Var(e) = \sigma^2 I_n$. Sometimes the prior information about the unknown regression parameters β are available as linear restrictions that can be given as follows:

$R\beta = r,$ (2)

where R is an $m \times p$ non zero matrix with rank$(R) = m < p$ and r is an $m \times 1$ vector. The RLS estimator is given by

$\hat{\beta}_{RLS} = \hat{\beta} + S^{-1}R'(RS^{-1}R')^{-1}(r - R\hat{\beta}),$ (3)

Where $S^{-1} = (X'X)^{-1}$and $\hat{\beta} = S^{-1}X'y$, the Ordinary Least Square estimator (OLSE). When the multicollinearity exists in the linear regression matrix, the result of the restricted least square estimator is unstable and misleading. Therefore, the restricted biased estimation as one of best to addressing of this problem there exist. So that, in order to improve the efficiency the RLS estimator, Kacıranlar [2], proposed the Restricted Liu estimator (RL) as follows:

$\hat{\beta}_{rd} = (X'X + I)^{-1}(X'X + dI)\hat{\beta}_{RLS}.$ (4)

But the RL estimator doses not satisfy the linear restrictions (2). Hua et al. [3] introduced restricted almost unbiased two parameter estimator (RAUTPE) based on the restricted two parameter estimator (RTPE) which proposed by Ozkale and Kaciranlar [4]. The RAUTPE estimator is denoted by $\hat{\beta}_{RAUTPE}(k, d)$ and it is given as follows

$\hat{\beta}_{RAUTPE}(k, d) = [I - (I - N_{kd}S)^2,$ (5)

where $N_{kd} = L_{kd}^{-1} - L_{kd}^{-1}R'(RL_{kd}^{-1}R')^{-1}RL_{kd}^{-1}$, and $L_{kd}^{-1} = (S + kI)^{-1}(I + kdS^{-1})$. We can observe that, the RAUTPE estimator satisfies the linear restrictions (2) when $R\beta = 0$. Bader and Mustafa [1] proposed the Shrinkage Restricted Ridge Regression Estimator (SRRE), by combining in a particular way the two approaches underlying the RLS and shrinkage of parameter the ridge regression. The SRRE is given as follows

$\hat{\beta}_{SRRE}(k) = \left(I - k\left(X'X + kI_p\right)^{-1}\hat{\beta}_{RLS}\right.$

$= (I - kS_k^{-1})\hat{\beta}_{RLS} \quad , \ k \geq 0$

$= M\hat{\beta}_{RLS},$ (6)

where $M = (I - kS_k^{-1})$, $S_k^{-1} = \left(X'X + kI_p\right)^{-1}$. Also, the SRRE estimator doses not satisfy the linear restrictions (2).

The goal of this paper is to review and compare the performance of some restricted estimators in order to determine which estimator has good statistical properties comparing with others. In section 2, we study the statistical properties of the RLS, RL, RAUTPE and SRRE estimators while in section 3, we make review and compare through simulation study the RLS, RLE, RAUTPE and SRRE estimators. Section 4 contains the numerical example to show the performance of these estimators. finall, the conclusion with some remarks are given in section 5.

2. Some Restricted Estimators and Its Properties

In this section, we want to show the statistical properties of the RLS, RLE, RAUTPE and SRRE estimators. The mean square error (MSE) of any estimators is given by:

$$MSE(\beta^*) = Var(\beta^*) + \left(bais(\beta^*)\right).\left(bias(\beta^*)\right)', \quad (7)$$

where

$$Var(\beta^*) = E[(\beta^* - E(\beta^*)((\beta^* - E(\beta^*))'] \quad (8)$$

and

$$Bias(\beta^*) = E(\beta^*) - \beta, \quad (9)$$

where $E(\beta^*)$ the expected value of β^*. The scalar mean square (mse) of any estimator is given as follows

$$mse(\beta^*) = tr\, Var(\beta^*) + \|E(\beta^*) - \beta\|^2, \quad (10)$$

where tr denote the trace of matrix.

2.1 Restricted Liu Estimator (RL)

The variance, bias, mean square error matrix (MSE) and (mse) of the RL estimator are given by

$$Var(\hat{\beta}_{rd}) = \sigma^2 F_d A F_d' \quad (11)$$

$$bias(\hat{\beta}_{rd}) = (d-1)(S+I)^{-1}\beta \quad (12)$$

$$MSE(\hat{\beta}_{rd}) = \sigma^2 F_d A F_d' + (d-1)^2 \beta'(S+I)^{-2}\beta, \quad (13)$$

Where $F_d = (S+I)^{-1}(S+dI)$. So that, the mse of RL estimator is given by

$$mse(\hat{\beta}_{rd}) = \sigma^2 tr(F_d A F_d') + (d-1)^2 \beta'^{(S+I)^{-2}}\beta. \quad (14)$$

2.2 Restricted Almost Unbiased Two Parameter Estimator (RAUTPE)

Hua et al.[3] proposed the RAUTPE. The statistical properties of the RAUTPE are given by:

$$Var\left(\hat{\beta}_{RAUTPE}(k,d)\right) = \sigma^2[I + k(1-d)N_{kd}(I+kdS^{-1})^{-1}] . N_{kd}SN_{kd}[I + k(1-d)N_{kd}(I+kdS^{-1})^{-1}] . \quad (15)$$

$$bias\left(\hat{\beta}_{RAUTPE}(k,d)\right) = -k^2(d-1)^2[N_{kd}(I+kdS^{-1})^{-1}]^2. \quad (16)$$

$$MSE\left(\hat{\beta}_{RAUTPE}(k,d)\right) = \sigma^2[I + k(1-d)N_{kd}(I+kdS^{-1})^{-1}] . N_{kd}SN_{kd}[I + k(1-d)N_{kd}(I+kdS^{-1})^{-1}] + k^4(d-1)^4\beta'[(I+kdS^{-1})^{-1}N_{kd}]^2\,[N_{kd}(I+kdS^{-1})^{-1}]^2. \quad (17)$$

Therefore, the mse of RAUTPE estimator as follows

$$mse\left(\hat{\beta}_{RAUTPE}(k,d)\right) = \sigma^2 tr[[I + k(1-d)N_{kd}(I+kdS^{-1})^{-1}] . N_{kd}SN_{kd}[I + k(1-d)N_{kd}(I+kdS^{-1})^{-1}]] + k^4(d-1)^4\beta'[(I+kdS^{-1})^{-1}N_{kd}]^2\,[N_{kd}(I+kdS^{-1})^{-1}]^2, \quad (18)$$

2.3 Shrinkage Restricted Ridge Regression Estimator (SRRE)

The variance, bias, mean square error matrix and scalar mean square error of the SRRE estimator are given by:

$$Var(\hat{\beta}_{SRRE}(k)) = \sigma^2 MAM' \quad (19)$$

$$Bias(\hat{\beta}_{SRRE}(k)) = -k^2 S_k^{-1}\beta \quad (20)$$

$$MSE(\hat{\beta}_{SRRE}(k)) = \sigma^2 MAM' + k^2 S_k^{-1}\beta\beta' S_k^{-1}, \quad (21)$$

where $= N_0 S N_0'$, $S = X'X$ and $N_0 = S^{-1} - S^{-1}H'(HS^{-1}H')^{-1}HS^{-1}$.

The mse of the SRRE estimator is given by

$$mse(\beta_{SRRE}(k)) = \sigma^2 tr(MAM') + k^2 tr(S_k^{-1}\beta'\beta S_k^{-1}) . \quad (22)$$

2.4.Estimated Ridge Parameter k

Hoerl and Kennard [5] showed the properties of Ordinary ridge regression in detail. They concluded that, the total variance decreases and the squared bias increases as k increases**.** The variance function is monotonically decreasing and the squared bias function is monotonically increasing. For this reason, there are many articles proposed different ridge parameters in the literature using different techniques. We use the MSE function to find out the performance of these estimators. Hoerl and Kennard [5], introduced k as follows:

$$k_{HK} = \frac{\hat{\sigma}^2}{\hat{\gamma}^2_{max}OLSE}, \quad (23)$$

where $\hat{\gamma}^2_{max}$ is the maximum element of $\hat{\gamma}_{OLSE}$. Hoerl [6], proposed k is denoted by:

$$k_{HKB} = \frac{p\hat{\sigma}^2}{\hat{\gamma}_{OLSE}'\hat{\gamma}_{OLSE}} . \quad (24)$$

-Lawless and Wang [7], suggested k as follows:

$$k_{LW} = \frac{p\hat{\sigma}^2}{\hat{\gamma}_{OLSE}'X'X\hat{\gamma}_{OLSE}}. \quad (25)$$

-Hocking [8], proposed k as follows:

$$k_{HSL} = \hat{\sigma}^2 \frac{(\sum_{i=1}^{p}(\lambda_i\hat{\gamma}_{OLSE})^2)}{\sum_{i=1}^{p}(\lambda_i\hat{\gamma}^2_{OLSE})^2} . \quad (26)$$

-Nomura [9], suggested k as follows:

$$k_{HMO} = \frac{p\hat{\sigma}^2}{\left(\sum_{i=1}^{p}\left(\frac{\gamma_{iOLSE}}{\left(1+\left(1+\lambda_i\left(\frac{\hat{\gamma}^2_{iOLSE}}{\hat{\sigma}^2}\right)^{\frac{1}{2}}\right)\right)}\right)\right)} . \quad (27)$$

-Kibria [10], suggested the estimators for k based on Arithmetic Mean (AM), Geometric Mean (GM), and median of $\frac{\hat{\sigma}^2}{\gamma_i^2}$. These are defined as follows:

The estimator based on AM is denoted by k_{AM} as the follows:

$$k_{AM} = \frac{1}{p}\sum_{i=1}^{p}\frac{\hat{\sigma}^2}{\hat{\gamma}_{iOLSE}}, \quad (28)$$

based on (GM), the estimator k_{GM} as the follows:

$$k_{GM} = \frac{\hat{\sigma}^2}{(\prod_{i=1}^{p}\hat{\gamma}^2_{OLSE})^{1/p}}, \quad (29)$$

based on median, the ridge parameter k_{MED} as follow:

$$k_{MED} = midian\left\{\frac{\hat{\sigma}^2}{\hat{\gamma}_{OLSE}^2}\right\}. \quad (30)$$

-Khalaf and Shukur [11], suggested based on k_{HK} the k_{KS} as:

$$k_{KS} = \frac{\lambda_{max}\hat{\sigma}^2}{(n-p)\hat{\sigma}^2+\lambda_{max}\hat{\gamma}_{max}^2 OLSE}, \quad (31)$$

Where λ_{max} the maximum eigenvalues of $X'X$.

-Alkhamisi [12], proposed the following estimators of k based on Kibria [13], Khalaf and Shukur [11] as:

$$k_{s_{airth}} = \frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\hat{\sigma}^2}{(n-p)\hat{\sigma}^2+\lambda_i\gamma_i^2 OLSE}. \quad (32)$$

$$k_{s_{md}} = midian\left(\frac{\lambda_i\hat{\sigma}^2}{(n-p)\hat{\sigma}^2+\lambda_i\hat{\gamma}_i^2 OLSE}\right). \quad (33)$$

Lateef and Alheety [14] , introduced the following estimators of k

$$k_{MU1} = \frac{\lambda_{med}\sum_{i=1}^{p}\hat{\gamma}_i^2 OLSE}{\lambda_{max}}. \quad (34)$$

$$k_{MU2} = \left|\frac{p\hat{\sigma}^2}{\hat{\gamma}_{OLSE}'\hat{\gamma}_{OLSE}} - \frac{p\hat{\sigma}^2}{\hat{\gamma}_{OLSE}'X'X\hat{\gamma}_{OLSE}}\right|. \quad (35)$$

$$k_{MU3} = min\left(\sqrt{\frac{\lambda_{min}\sum_{i=1}^{p}\hat{\gamma}_i^2 OLSE}{\hat{\sigma}^2}}\right). \quad (36)$$

$$k_{MU4} = max\left(\sqrt{\frac{\lambda_{min}\sum_{i=1}^{p}\hat{\gamma}_i^2 OLSE}{\hat{\sigma}^2}}\right). \quad (37)$$

3. Simulation Study

In this section, we make simulation study of the RLS, RL, RAUTPE, and SRRE estimators by using the Matlab program. This simulation is created depending on factors that affect the properties of the estimator's duo to the degree of the collinearity among several explanatory variables. Kibria [13], was followed to generate the explanatory variables by using the equation.

$$x_{ij} = (1-\emptyset^2)^{1/2} z_{ij} + \emptyset z_{ip}, \ i = 1,2,\dots n, j = 1,2,..p \quad (38)$$

where the z_{ij} independent standard normal pseudo-random numbers and $\emptyset$ represents the correlation between any two variables. These variables are standardized so that $X'X$ is being in correlation form. The response variable y is considered by:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots \dots \dots \dots + \beta_p x_{ip} + e_i,$$
$$i = 1,2,\dots,n \quad (39)$$

where e_i is i.i.d. $N(0,\sigma^2)$. Therefore, zero intercept for (39) will be assumed. Also the number of explanatory variables $p = 4$, the values of the standard deviation σ are choosen as (0.1 , 1 , 5 , 15) The correlation coefficient $\emptyset$ will choose as (0.85,0.95, 0.99) and sample size n as (50, 100 ,150,).The coefficients $\beta_1, \beta_2, \dots, \beta_p$ are selected as the eigenvectors corresponding to the largest eigenvalue of the matrix $X'X$ subject to constraint $\beta'\beta$ = 1. Thus, for all $n, \sigma, \lambda, p, \beta$ and $\emptyset$, sets of Xs are created. The experiment was replicated 10000 times by creating new error terms. Estimated mean square error (EMSE) is calculated as follows:

$$EMSE(\beta^*) = \frac{1}{10000}\sum_{i=1}^{10000}(\beta^*-\beta)'(\beta^*-\beta),$$

where β^* would be any estimators (RLS , RL ,RAUTPE or SRRE).

Table 1: Estimated MSE when $n = 50$, $\emptyset = .85$, $p = 4$

σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}	σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}
0.1	k_{HK}	1.1228	0.4697	0.4654	0.4692	1	k_{HK}	0.7882	0.3041	0.3082	0.2597
	k_{HKB}	1.1228	0.4697	0.4654	0.4677		k_{HKB}	0.7882	0.3041	0.3082	0.2100
	k_{LW}	1.1228	0.4697	0.4654	0.4692		k_{LW}	0.7882	0.3041	0.3082	0.2630
	k_{HSL}	1.1228	0.4697	0.4654	0.3680		k_{HSL}	0.7882	0.3041	0.3082	0.1590
	k_{HMO}	1.1228	0.4697	0.4654	0.4454		k_{HMO}	0.7882	0.3041	0.3082	0.1825
	k_{AM}	1.1228	0.4697	0.4654	0.4278		k_{AM}	0.7882	0.3041	0.3082	0.1583
	k_{GM}	1.1228	0.4697	0.4654	0.4692		k_{GM}	0.7882	0.3041	0.3082	0.2691
	k_{MED}	1.1228	0.4697	0.4654	0.4607		k_{MED}	0.7882	0.3041	0.3082	0.2870
	k_{KS}	1.1228	0.4697	0.4654	0.4554		k_{KS}	0.7882	0.3041	0.3082	0.2691
	$k_{S_{arith}}$	1.1228	0.4697	0.4654	0.4592		$k_{S_{arith}}$	0.7882	0.3041	0.3082	0.2908
	k_{SMD}	1.1228	0.4697	0.4654	0.4643		k_{SMD}	0.7882	0.3041	0.3082	0.2899
	k_{MU1}	1.1228	0.4697	0.4654	0.2359		k_{MU1}	0.7882	0.3041	0.3082	0.0767
	k_{MU2}	1.1228	0.4697	0.4654	0.2359		k_{MU2}	0.7882	0.3041	0.3082	0.0768
	k_{MU3}	1.1228	0.4697	0.4654	0.4554		k_{MU3}	0.7882	0.3041	0.3082	0.2691
	k_{MU4}	1.1228	0.4697	0.4654	0.4643		k_{MU4}	0.7882	0.3041	0.3082	0.2899

Table 2: Estimated MSE when $n = 50$, $\emptyset = .85$, $p = 4$

σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}	σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}
5	k_{HK}	1.0559	0.4920	0.8984	0.3429	15	k_{HK}	1.0477	0.3230	0.3237	0.2991
	k_{HKB}	1.0559	0.4923	0.8984	0.3429		k_{HKB}	1.0477	0.3230	0.3237	0.2719
	k_{LW}	1.0559	0.4921	0.8984	0.3429		k_{LW}	1.0477	0.3230	0.3237	0.2678
	k_{HSL}	1.0559	0.4926	0.8984	0.3537		k_{HSL}	1.0477	0.3230	0.3241	0.4983
	k_{HMO}	1.0559	0.4925	0.8984	0.3453		k_{HMO}	1.0477	0.3230	0.3237	0.4089
	k_{AM}	1.0559	0.4924	0.8984	0.3522		k_{AM}	1.0477	0.3230	0.3238	0.4493
	k_{GM}	1.0559	0.4918	0.8984	0.3429		k_{GM}	1.0477	0.3230	0.3237	0.3058
	k_{MED}	1.0559	0.4917	0.8984	0.3435		k_{MED}	1.0477	0.3230	0.3237	0.3081
	k_{KS}	1.0559	0.4919	0.8984	0.3440		k_{KS}	1.0477	0.3230	0.3237	0.2867
	$k_{S_{arith}}$	1.0559	0.4917	0.8984	0.3435		$k_{S_{arith}}$	1.0477	0.3230	0.3237	0.3189
	k_{SMD}	1.0559	0.4917	0.8984	0.3432		k_{SMD}	1.0477	0.3230	0.3237	0.3162
	k_{MU1}	1.0559	0.4952	0.8984	0.4126		k_{MU1}	1.0477	0.3230	0.3240	0.4800
	k_{MU2}	1.0559	0.4946	0.8984	0.4032		k_{MU2}	1.0477	0.3230	0.3237	0.2779
	k_{MU3}	1.0559	0.4919	0.8984	0.3440		k_{MU3}	1.0477	0.3230	0.3237	0.2867
	k_{MU4}	1.0559	0.4917	0.8984	0.3432		k_{MU4}	1.0477	0.3230	0.3237	0.3162

Table3: Estimated MSE when $n = 50$, $\emptyset = .95$, $p = 4$

σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}	σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}
0.1	k_{HK}	0.9732	0.8957	0.9225	0.9192	1	k_{HK}	0.5100	0.4919	0.4859	0.3658
	k_{HKB}	0.9732	0.8957	0.9225	0.9093		k_{HKB}	0.5100	0.4919	0.4859	0.3295
	k_{LW}	0.9732	0.8957	0.9225	0.9192		k_{LW}	0.5100	0.4919	0.4859	0.3183
	k_{HSL}	0.9732	0.8957	0.9221	0.4005		k_{HSL}	0.5100	0.4919	0.4859	0.3153
	k_{HMO}	0.9732	0.8957	0.9225	0.2853		k_{HMO}	0.5100	0.4919	0.4859	0.3175
	k_{AM}	0.9732	0.8957	0.9225	0.2058		k_{AM}	0.5100	0.4919	0.4859	0.3176
	k_{GM}	0.9732	0.8957	0.9225	0.9192		k_{GM}	0.5100	0.4919	0.4859	0.3966
	k_{MED}	0.9732	0.8957	0.9225	0.8939		k_{MED}	0.5100	0.4919	0.4859	0.4300
	k_{KS}	0.9732	0.8957	0.9225	0.8750		k_{KS}	0.5100	0.4919	0.4859	0.3606
	$k_{S_{arith}}$	0.9732	0.8957	0.9225	0.8912		$k_{S_{arith}}$	0.5100	0.4919	0.4859	0.4813
	k_{SMD}	0.9732	0.8957	0.9225	0.9025		k_{SMD}	0.5100	0.4919	0.4859	0.4712
	k_{MU1}	0.9732	0.8957	0.9222	0.3470		k_{MU1}	0.5100	0.4919	0.4859	0.3337
	k_{MU2}	0.9732	0.8957	0.9225	0.1781		k_{MU2}	0.5100	0.4919	0.4859	0.3265
	k_{MU3}	0.9732	0.8957	0.9225	0.8750		k_{MU3}	0.5100	0.4919	0.4859	0.3606
	k_{MU4}	0.9732	0.8957	0.9225	0.9025		k_{MU4}	0.5100	0.4919	0.4859	0.4712

Table 4: Estimated MSE when $n = 50$, $\emptyset = .95$, $p = 4$

σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}	σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}
	k_{HK}	0.9277	0.3169	0.3151	0.3058		k_{HK}	0.8767	0.1644	0.1635	0.1361
	k_{HKB}	0.9277	0.3169	0.3151	0.2993		k_{HKB}	0.8767	0.1644	0.1635	0.1211
	k_{LW}	0.9277	0.3169	0.3151	0.2990		k_{LW}	0.8767	0.1644	0.1635	0.1193
	k_{HSL}	0.9277	0.3169	0.3152	0.3311		k_{HSL}	0.8767	0.1644	0.1636	0.1321
	k_{HMO}	0.9277	0.3169	0.3151	0.3035		k_{HMO}	0.8767	0.1644	0.1635	0.1074
	k_{AM}	0.9277	0.3169	0.3151	0.3033		k_{AM}	0.8767	0.1644	0.1635	0.1157
	k_{GM}	0.9277	0.3169	0.3151	0.3073		k_{GM}	0.8767	0.1644	0.1635	0.1410
	k_{MED}	0.9277	0.3169	0.3151	0.3067		k_{MED}	0.8767	0.1644	0.1635	0.1394
5	k_{KS}	0.9277	0.3169	0.3151	0.2996	15	k_{KS}	0.8767	0.1644	0.1635	0.1143
	$k_{S_{arith}}$	0.9277	0.3169	0.3151	0.3140		$k_{S_{arith}}$	0.8767	0.1644	0.1635	0.1597
	k_{SMD}	0.9277	0.3169	0.3151	0.3126		k_{SMD}	0.8767	0.1644	0.1635	0.1558
	k_{MU1}	0.9277	0.3169	0.3152	0.3540		k_{MU1}	0.8767	0.1644	0.1636	0.1784
	k_{MU2}	0.9277	0.3169	0.3151	0.3142		k_{MU2}	0.8767	0.1644	0.1635	0.1110
	k_{MU3}	0.9277	0.3169	0.3151	0.2996		k_{MU3}	0.8767	0.1644	0.1635	0.1143
	k_{MU4}	0.9277	0.3169	0.3151	0.3126		k_{MU4}	0.8767	0.1644	0.1635	0.1558

Table 5: Estimated MSE when $n = 50$, $\emptyset = .99$, $p = 4$

σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}	σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}
	k_{HK}	1.0917	0.9481	0.9394	0.9275		k_{HK}	1.4123	0.5169	0.5244	0.3091
	k_{HKB}	1.0917	0.9481	0.9394	0.8747		k_{HKB}	1.4123	0.5169	0.5244	0.1759
	k_{LW}	1.0917	0.9481	0.9394	0.9276		k_{LW}	1.4123	0.5169	0.5244	0.1043
	k_{HSL}	1.0917	0.9481	0.9394	0.3260		k_{HSL}	1.4123	0.5169	0.5244	0.0542
	k_{HMO}	1.0917	0.9481	0.9394	0.5996		k_{HMO}	1.4123	0.5169	0.5244	0.1049
	k_{AM}	1.0917	0.9481	0.9394	0.3477		k_{AM}	1.4123	0.5169	0.5244	0.1179
	k_{GM}	1.0917	0.9481	0.9394	0.9276		k_{GM}	1.4123	0.5169	0.5244	0.3194
	k_{MED}	1.0917	0.9481	0.9394	0.9142		k_{MED}	1.4123	0.5169	0.5244	0.3050
	k_{KS}	1.0917	0.9481	0.9394	0.9040		k_{KS}	1.4123	0.5169	0.5244	0.1302
0.1	$k_{S_{arith}}$	1.0917	0.9481	0.9394	0.9127	1	$k_{S_{arith}}$	1.4123	0.5169	0.5244	0.4989
	k_{SMD}	1.0917	0.9481	0.9394	0.9155		k_{SMD}	1.4123	0.5169	0.5244	0.4666
	k_{MU1}	1.0917	0.9481	0.9394	0.0879		k_{MU1}	1.4123	0.5169	0.5244	0.0104
	k_{MU2}	1.0917	0.9481	0.9394	0.0900		k_{MU2}	1.4123	0.5169	0.5244	0.0197
	k_{MU3}	1.0917	0.9481	0.9394	0.9040		k_{MU3}	1.4123	0.5169	0.5244	0.1302
	k_{MU4}	1.0917	0.9481	0.9394	0.9155		k_{MU4}	1.4123	0.5169	0.5244	0.4666

Table 6: Estimated MSE when $n = 50$, $\emptyset = .99$, $p = 4$

σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}	σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}
	k_{HK}	0.5496	0.2410	0.2403	0.2176		k_{HK}	1.7427	0.5679	0.5755	0.3719
	k_{HKB}	0.5496	0.2410	0.2403	0.2134		k_{HKB}	1.7427	0.5679	0.5755	0.2052
	k_{LW}	0.5496	0.2410	0.2403	0.2299		k_{LW}	1.7427	0.5679	0.5755	0.0227
	k_{HSL}	0.5496	0.2410	0.2404	0.4086		k_{HSL}	1.7427	0.5679	0.5755	0.0061
	k_{HMO}	0.5496	0.2410	0.2403	0.2490		k_{HMO}	1.7427	0.5679	0.5755	0.0628
	k_{AM}	0.5496	0.2410	0.2404	0.3707		k_{AM}	1.7427	0.5679	0.5755	0.1202
	k_{GM}	0.5496	0.2410	0.2403	0.2193		k_{GM}	1.7427	0.5679	0.5755	0.3776
	k_{MED}	0.5496	0.2410	0.2403	0.2175		k_{MED}	1.7427	0.5679	0.5755	0.1155
5	k_{KS}	0.5496	0.2410	0.2403	0.2137	15	k_{KS}	1.7427	0.5679	0.5755	0.0369
	$k_{S_{arith}}$	0.5496	0.2410	0.2403	0.2388		$k_{S_{arith}}$	1.7427	0.5679	0.5755	0.5496
	k_{SMD}	0.5496	0.2410	0.2403	0.2357		k_{SMD}	1.7427	0.5679	0.5755	0.4916
	k_{MU1}	0.5496	0.2410	0.2404	0.3627		k_{MU1}	1.7427	0.5679	0.5755	0.0032
	k_{MU2}	0.5496	0.2410	0.2403	0.2297		k_{MU2}	1.7427	0.5679	0.5755	0.0223
	k_{MU3}	0.5496	0.2410	0.2403	0.2137		k_{MU3}	1.7427	0.5679	0.5755	0.0369
	k_{MU4}	0.5496	0.2410	0.2403	0.2357		k_{MU4}	1.7427	0.5679	0.5755	0.4916

Table 7: Estimated MSE when $n = 100$, $\emptyset = .85$, $p = 4$

σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}	σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}
	k_{HK}	0.6166	0.4813	0.4815	0.4812		k_{HK}	0.5308	0.4638	0.4637	0.4498
	k_{HKB}	0.6166	0.4813	0.4815	0.4803		k_{HKB}	0.5308	0.4638	0.4637	0.4161
	k_{LW}	0.6166	0.4813	0.4815	0.4812		k_{LW}	0.5308	0.4638	0.4637	0.4499
	k_{HSL}	0.6166	0.4813	0.4815	0.3735		k_{HSL}	0.5308	0.4638	0.4637	0.3877
	k_{HMO}	0.6166	0.4813	0.4815	0.4469		k_{HMO}	0.5308	0.4638	0.4637	0.2912
	k_{AM}	0.6166	0.4813	0.4815	0.3631		k_{AM}	0.5308	0.4638	0.4637	0.3106
	k_{GM}	0.6166	0.4813	0.4815	0.4812		k_{GM}	0.5308	0.4638	0.4637	0.4531
	k_{MED}	0.6166	0.4813	0.4815	0.4766		k_{MED}	0.5308	0.4638	0.4637	0.4574
0.1	k_{KS}	0.6166	0.4813	0.4815	0.4741	1	k_{KS}	0.5308	0.4638	0.4637	0.4531
	$k_{S_{arith}}$	0.6166	0.4813	0.4815	0.4755		$k_{S_{arith}}$	0.5308	0.4638	0.4637	0.4586
	k_{SMD}	0.6166	0.4813	0.4815	0.4785		k_{SMD}	0.5308	0.4638	0.4637	0.4579
	k_{MU1}	0.6166	0.4813	0.4815	0.2583		k_{MU1}	0.5308	0.4638	0.4637	0.3735
	k_{MU2}	0.6166	0.4813	0.4815	0.2529		k_{MU2}	0.5308	0.4638	0.4637	0.3289
	k_{MU3}	0.6166	0.4813	0.4815	0.4741		k_{MU3}	0.5308	0.4638	0.4637	0.4531
	k_{MU4}	0.6166	0.4813	0.4815	0.4785		k_{MU4}	0.5308	0.4638	0.4637	0.4579

Table 8: Estimated MSE when $n = 100$, $\emptyset = .85$, $p = 4$

σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}	σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}
5	k_{HK}	0.4599	0.3305	0.3304	0.3294	15	k_{HK}	0.4529	0.4557	0.4557	0.4370
	k_{HKB}	0.4599	0.3305	0.3304	0.3318		k_{HKB}	0.4529	0.4557	0.4557	0.4369
	k_{LW}	0.4599	0.3305	0.3304	0.3281		k_{LW}	0.4529	0.4557	0.4557	0.4361
	k_{HSL}	0.4599	0.3305	0.3304	0.3379		k_{HSL}	0.4529	0.4557	0.4557	0.4416
	k_{HMO}	0.4599	0.3305	0.3304	0.3337		k_{HMO}	0.4529	0.4557	0.4557	0.4388
	k_{AM}	0.4599	0.3305	0.3304	0.3303		k_{AM}	0.4529	0.4557	0.4557	0.4390
	k_{GM}	0.4599	0.3305	0.3304	0.3289		k_{GM}	0.4529	0.4557	0.4557	0.4497
	k_{MED}	0.4599	0.3305	0.3304	0.3298		k_{MED}	0.4529	0.4557	0.4557	0.4534
	k_{KS}	0.4599	0.3305	0.3304	0.3289		k_{KS}	0.4529	0.4557	0.4557	0.4492
	$k_{S_{arith}}$	0.4599	0.3305	0.3304	0.3302		$k_{S_{arith}}$	0.4529	0.4557	0.4557	0.4550
	k_{SMD}	0.4599	0.3305	0.3304	0.3300		k_{SMD}	0.4529	0.4557	0.4557	0.4545
	k_{MU1}	0.4599	0.3305	0.3304	0.3582		k_{MU1}	0.4529	0.4557	0.4557	0.4473
	k_{MU2}	0.4599	0.3305	0.3304	0.3457		k_{MU2}	0.4529	0.4557	0.4557	0.4433
	k_{MU3}	0.4599	0.3305	0.3304	0.3289		k_{MU3}	0.4529	0.4557	0.4557	0.4492
	k_{MU4}	0.4599	0.3305	0.3304	0.3300		k_{MU4}	0.4529	0.4557	0.4557	0.4545

Table 9: Estimated MSE when $n = 100$, $\emptyset = .95$, $p = 4$

σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}	σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}
0.1	k_{HK}	0.9277	0.8051	0.8061	0.8046	1	k_{HK}	0.8018	0.4851	0.4860	0.3793
	k_{HKB}	0.9277	0.8051	0.8061	0.8002		k_{HKB}	0.8018	0.4851	0.4860	0.3312
	k_{LW}	0.9277	0.8051	0.8061	0.8046		k_{LW}	0.8018	0.4851	0.4860	0.3852
	k_{HSL}	0.9277	0.8051	0.8061	0.2399		k_{HSL}	0.8018	0.4851	0.4860	0.3133
	k_{HMO}	0.9277	0.8051	0.8061	0.4776		k_{HMO}	0.8018	0.4851	0.4860	0.3222
	k_{AM}	0.9277	0.8051	0.8061	0.4753		k_{AM}	0.8018	0.4851	0.4860	0.3190
	k_{GM}	0.9277	0.8051	0.8061	0.8046		k_{GM}	0.8018	0.4851	0.4860	0.4014
	k_{MED}	0.9277	0.8051	0.8061	0.7944		k_{MED}	0.8018	0.4851	0.4860	0.4549
	k_{KS}	0.9277	0.8051	0.8061	0.7893		k_{KS}	0.8018	0.4851	0.4860	0.4014
	$k_{S_{arith}}$	0.9277	0.8051	0.8061	0.7919		$k_{S_{arith}}$	0.8018	0.4851	0.4860	0.4742
	k_{SMD}	0.9277	0.8051	0.8061	0.7977		k_{SMD}	0.8018	0.4851	0.4860	0.4685
	k_{MU1}	0.9277	0.8051	0.8061	0.2543		k_{MU1}	0.8018	0.4851	0.4860	0.1055
	k_{MU2}	0.9277	0.8051	0.8061	0.1536		k_{MU2}	0.8018	0.4851	0.4860	0.0935
	k_{MU3}	0.9277	0.8051	0.8061	0.7893		k_{MU3}	0.8018	0.4851	0.4860	0.4014
	k_{MU4}	0.9277	0.8051	0.8061	0.7977		k_{MU4}	0.8018	0.4851	0.4860	0.4685

Table 10: Estimated MSE when $n = 100$, $\emptyset = .95$, $p = 4$

σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}	σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}
5	k_{HK}	0.5259	0.7049	0.7043	0.5245	15	k_{HK}	0.5293	0.2873	0.2871	0.2767
	k_{HKB}	0.5259	0.7049	0.7043	0.5116		k_{HKB}	0.5293	0.2873	0.2871	0.2793
	k_{LW}	0.5259	0.7049	0.7043	0.5104		k_{LW}	0.5293	0.2873	0.2871	0.2821
	k_{HSL}	0.5259	0.7049	0.7043	0.5096		k_{HSL}	0.5293	0.2873	0.2871	0.3564
	k_{HMO}	0.5259	0.7049	0.7043	0.5105		k_{HMO}	0.5293	0.2873	0.2871	0.2904
	k_{AM}	0.5259	0.7049	0.7043	0.5087		k_{AM}	0.5293	0.2873	0.2871	0.2798
	k_{GM}	0.5259	0.7049	0.7043	0.5951		k_{GM}	0.5293	0.2873	0.2871	0.2803
	k_{MED}	0.5259	0.7049	0.7043	0.6574		k_{MED}	0.5293	0.2873	0.2871	0.2840
	k_{KS}	0.5259	0.7049	0.7043	0.5872		k_{KS}	0.5293	0.2873	0.2871	0.2792
	$k_{S_{arith}}$	0.5259	0.7049	0.7043	0.7000		$k_{S_{arith}}$	0.5293	0.2873	0.2871	0.2867
	k_{SMD}	0.5259	0.7049	0.7043	0.6935		k_{SMD}	0.5293	0.2873	0.2871	0.2862
	k_{MU1}	0.5259	0.7049	0.7043	0.4997		k_{MU1}	0.5293	0.2873	0.2871	0.3656
	k_{MU2}	0.5259	0.7049	0.7043	0.4997		k_{MU2}	0.5293	0.2873	0.2871	0.2966
	k_{MU3}	0.5259	0.7049	0.7043	0.5872		k_{MU3}	0.5293	0.2873	0.2871	0.2792
	k_{MU4}	0.5259	0.7049	0.7043	0.6935		k_{MU4}	0.5293	0.2873	0.2871	0.2862

Table 11: Estimated MSE when $n = 100$, $\emptyset = .99$, $p = 4$

σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}	σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}
0.1	k_{HK}	0.8767	0.7899	0.7944	0.7869	1	k_{HK}	0.8767	0.4936	0.4914	0.2365
	k_{HKB}	0.8767	0.7899	0.7944	0.7659		k_{HKB}	0.8767	0.4936	0.4914	0.1730
	k_{LW}	0.8767	0.7899	0.7944	0.7869		k_{LW}	0.8767	0.4936	0.4914	0.1552
	k_{HSL}	0.8767	0.7899	0.7944	0.0671		k_{HSL}	0.8767	0.4936	0.4914	0.1502
	k_{HMO}	0.8767	0.7899	0.7944	0.2890		k_{HMO}	0.8767	0.4936	0.4914	0.1470
	k_{AM}	0.8767	0.7899	0.7944	0.2770		k_{AM}	0.8767	0.4936	0.4914	0.1491
	k_{GM}	0.8767	0.7899	0.7944	0.7869		k_{GM}	0.8767	0.4936	0.4914	0.2439
	k_{MED}	0.8767	0.7899	0.7944	0.7793		k_{MED}	0.8767	0.4936	0.4914	0.2174
	k_{KS}	0.8767	0.7899	0.7944	0.7725		k_{KS}	0.8767	0.4936	0.4914	0.1574
	$k_{S_{arith}}$	0.8767	0.7899	0.7944	0.7790		$k_{S_{arith}}$	0.8767	0.4936	0.4914	0.3128
	k_{SMD}	0.8767	0.7899	0.7944	0.7804		k_{SMD}	0.8767	0.4936	0.4914	0.3034
	k_{MU1}	0.8767	0.7899	0.7944	0.1106		k_{MU1}	0.8767	0.4936	0.4914	0.1948
	k_{MU2}	0.8767	0.7899	0.7944	0.0337		k_{MU2}	0.8767	0.4936	0.4914	0.1585
	k_{MU3}	0.8767	0.7899	0.7944	0.7725		k_{MU3}	0.8767	0.4936	0.4914	0.1574
	k_{MU4}	0.8767	0.7899	0.7944	0.7804		k_{MU4}	0.8767	0.4936	0.4914	0.3034

Table 12: Estimated MSE when $n = 100$, $\emptyset = .99$, $p = 4$

σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}	σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}
	k_{HK}	0.5551	0.2674	0.2678	0.2174		k_{HK}	0.4729	0.2780	0.2778	0.1826
	k_{HKB}	0.5551	0.2674	0.2678	0.1889		k_{HKB}	0.4729	0.2780	0.2778	0.1278
	k_{LW}	0.5551	0.2674	0.2678	0.1303		k_{LW}	0.4729	0.2780	0.2778	0.1854
	k_{HSL}	0.5551	0.2674	0.2678	0.1300		k_{HSL}	0.4729	0.2780	0.2778	0.0947
	k_{HMO}	0.5551	0.2674	0.2678	0.1447		k_{HMO}	0.4729	0.2780	0.2778	0.1102
	k_{AM}	0.5551	0.2674	0.2678	0.1310		k_{AM}	0.4729	0.2780	0.2778	0.0982
	k_{GM}	0.5551	0.2674	0.2678	0.2212		k_{GM}	0.4729	0.2780	0.2778	0.2228
5	k_{MED}	0.5551	0.2674	0.2678	0.1942	15	k_{MED}	0.4729	0.2780	0.2778	0.3785
	k_{KS}	0.5551	0.2674	0.2678	0.1435		k_{KS}	0.4729	0.2780	0.2778	0.2228
	$k_{S_{arith}}$	0.5551	0.2674	0.2678	0.2644		$k_{S_{arith}}$	0.4729	0.2780	0.2778	0.4790
	k_{SMD}	0.5551	0.2674	0.2678	0.2572		k_{SMD}	0.4729	0.2780	0.2778	0.4644
	k_{MU1}	0.5551	0.2674	0.2678	0.1507		k_{MU1}	0.4729	0.2780	0.2778	0.0410
	k_{MU2}	0.5551	0.2674	0.2678	0.1324		k_{MU2}	0.4729	0.2780	0.2778	0.0455
	k_{MU3}	0.5551	0.2674	0.2678	0.1435		k_{MU3}	0.4729	0.2780	0.2778	0.2228
	k_{MU4}	0.5551	0.2674	0.2678	0.2572		k_{MU4}	0.4729	0.2780	0.2778	0.4644

Table 13: Estimated MSE when $n = 150$, $\emptyset = .85$, $p = 4$

σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}	σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}
	k_{HK}	0.6777	0.5295	0.5297	0.5295		k_{HK}	0.6202	0.5041	0.5040	0.4855
	k_{HKB}	0.6777	0.5295	0.5297	0.5290		k_{HKB}	0.6202	0.5041	0.5040	0.4409
	k_{LW}	0.6777	0.5295	0.5297	0.5295		k_{LW}	0.6202	0.5041	0.5040	0.4856
	k_{HSL}	0.6777	0.5295	0.5297	0.2229		k_{HSL}	0.6202	0.5041	0.5040	0.4850
	k_{HMO}	0.6777	0.5295	0.5297	0.4602		k_{HMO}	0.6202	0.5041	0.5040	0.2431
	k_{AM}	0.6777	0.5295	0.5297	0.3392		k_{AM}	0.6202	0.5041	0.5040	0.2532
	k_{GM}	0.6777	0.5295	0.5297	0.5295		k_{GM}	0.6202	0.5041	0.5040	0.4902
0.1	k_{MED}	0.6777	0.5295	0.5297	0.5257	1	k_{MED}	0.6202	0.5041	0.5040	0.4972
	k_{KS}	0.6777	0.5295	0.5297	0.5240		k_{KS}	0.6202	0.5041	0.5040	0.4902
	$k_{S_{arith}}$	0.6777	0.5295	0.5297	0.5246		$k_{S_{arith}}$	0.6202	0.5041	0.5040	0.4993
	k_{SMD}	0.6777	0.5295	0.5297	0.5274		k_{SMD}	0.6202	0.5041	0.5040	0.4981
	k_{MU1}	0.6777	0.5295	0.5297	0.2195		k_{MU1}	0.6202	0.5041	0.5040	0.4405
	k_{MU2}	0.6777	0.5295	0.5297	0.2114		k_{MU2}	0.6202	0.5041	0.5040	0.3319
	k_{MU3}	0.6777	0.5295	0.5297	0.5240		k_{MU3}	0.6202	0.5041	0.5040	0.4902
	k_{MU4}	0.6777	0.5295	0.5297	0.5274		k_{MU4}	0.6202	0.5041	0.5040	0.4981

Table 14: Estimated MSE when $n = 150$, $\emptyset = .85$, $p = 4$

σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}	σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}
5	k_{HK}	0.4596	0.3774	0.3774	0.3375	15	k_{HK}	0.4575	0.3115	0.3115	0.3022
	k_{HKB}	0.4596	0.3774	0.3774	0.3301		k_{HKB}	0.4575	0.3115	0.3115	0.2932
	k_{LW}	0.4596	0.3774	0.3774	0.3403		k_{LW}	0.4575	0.3115	0.3115	0.2947
	k_{HSL}	0.4596	0.3774	0.3774	0.3392		k_{HSL}	0.4575	0.3115	0.3115	0.3991
	k_{HMO}	0.4596	0.3774	0.3774	0.3291		k_{HMO}	0.4575	0.3115	0.3115	0.3005
	k_{AM}	0.4596	0.3774	0.3774	0.3298		k_{AM}	0.4575	0.3115	0.3115	0.2940
	k_{GM}	0.4596	0.3774	0.3774	0.3675		k_{GM}	0.4575	0.3115	0.3115	0.3076
	k_{MED}	0.4596	0.3774	0.3774	0.3740		k_{MED}	0.4575	0.3115	0.3115	0.3096
	k_{KS}	0.4596	0.3774	0.3774	0.3675		k_{KS}	0.4575	0.3115	0.3115	0.3062
	$k_{S_{arith}}$	0.4596	0.3774	0.3774	0.3762		$k_{S_{arith}}$	0.4575	0.3115	0.3115	0.3109
	k_{SMD}	0.4596	0.3774	0.3774	0.3755		k_{SMD}	0.4575	0.3115	0.3115	0.3105
	k_{MU1}	0.4596	0.3774	0.3774	0.3613		k_{MU1}	0.4575	0.3115	0.3115	0.3921
	k_{MU2}	0.4596	0.3774	0.3774	0.3334		k_{MU2}	0.4575	0.3115	0.3115	0.3046
	k_{MU3}	0.4596	0.3774	0.3774	0.3675		k_{MU3}	0.4575	0.3115	0.3115	0.3062
	k_{MU4}	0.4596	0.3774	0.3774	0.3755		k_{MU4}	0.4575	0.3115	0.3115	0.3105

Table 15: Estimated MSE when $n = 150$, $\emptyset = .95$, $p = 4$

σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}	σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}
0.1	k_{HK}	0.8741	0.6356	0.6361	0.6354	1	k_{HK}	0.8444	0.8208	0.8216	0.7321
	k_{HKB}	0.8741	0.6356	0.6361	0.6333		k_{HKB}	0.8444	0.8208	0.8216	0.5817
	k_{LW}	0.8741	0.6356	0.6361	0.6354		k_{LW}	0.8444	0.8208	0.8216	0.7328
	k_{HSL}	0.8741	0.6356	0.6361	0.1119		k_{HSL}	0.8444	0.8208	0.8216	0.4343
	k_{HMO}	0.8741	0.6356	0.6361	0.3108		k_{HMO}	0.8444	0.8208	0.8216	0.2323
	k_{AM}	0.8741	0.6356	0.6361	0.1077		k_{AM}	0.8444	0.8208	0.8216	0.2903
	k_{GM}	0.8741	0.6356	0.6361	0.6354		k_{GM}	0.8444	0.8208	0.8216	0.7503
	k_{MED}	0.8741	0.6356	0.6361	0.6297		k_{MED}	0.8444	0.8208	0.8216	0.7959
	k_{KS}	0.8741	0.6356	0.6361	0.6267		k_{KS}	0.8444	0.8208	0.8216	0.7503
	$k_{S_{arith}}$	0.8741	0.6356	0.6361	0.6284		$k_{S_{arith}}$	0.8444	0.8208	0.8216	0.8117
	k_{SMD}	0.8741	0.6356	0.6361	0.6317		k_{SMD}	0.8444	0.8208	0.8216	0.8061
	k_{MU1}	0.8741	0.6356	0.6361	0.1415		k_{MU1}	0.8444	0.8208	0.8216	0.3840
	k_{MU2}	0.8741	0.6356	0.6361	0.1079		k_{MU2}	0.8444	0.8208	0.8216	0.2204
	k_{MU3}	0.8741	0.6356	0.6361	0.6267		k_{MU3}	0.8444	0.8208	0.8216	0.7503
	k_{MU4}	0.8741	0.6356	0.6361	0.6317		k_{MU4}	0.8444	0.8208	0.8216	0.8061

Table 16: Estimated MSE when $n = 150$, $\emptyset = .95$, $p = 4$

σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}	σ	k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}
	k_{HK}	0.4518	0.4654	0.4654	0.3983		k_{HK}	0.6447	0.2206	0.2205	0.1986
	k_{HKB}	0.4518	0.4654	0.4654	0.3982		k_{HKB}	0.6447	0.2206	0.2205	0.1749
	k_{LW}	0.4518	0.4654	0.4654	0.3985		k_{LW}	0.6447	0.2206	0.2205	0.1669
	k_{HSL}	0.4518	0.4654	0.4654	0.4129		k_{HSL}	0.6447	0.2206	0.2205	0.2176
	k_{HMO}	0.4518	0.4654	0.4654	0.4008		k_{HMO}	0.6447	0.2206	0.2205	0.1751
	k_{AM}	0.4518	0.4654	0.4654	0.3979		k_{AM}	0.6447	0.2206	0.2205	0.2098
	k_{GM}	0.4518	0.4654	0.4654	0.4339		k_{GM}	0.6447	0.2206	0.2205	0.2058
5	k_{MED}	0.4518	0.4654	0.4654	0.4543	15	k_{MED}	0.6447	0.2206	0.2205	0.2101
	k_{KS}	0.4518	0.4654	0.4654	0.4339		k_{KS}	0.6447	0.2206	0.2205	0.1914
	k_{Sarith}	0.4518	0.4654	0.4654	0.4642		k_{Sarith}	0.6447	0.2206	0.2205	0.2196
	k_{SMD}	0.4518	0.4654	0.4654	0.4627		k_{SMD}	0.6447	0.2206	0.2205	0.2182
	k_{MU1}	0.4518	0.4654	0.4654	0.4245		k_{MU1}	0.6447	0.2206	0.2205	0.2352
	k_{MU2}	0.4518	0.4654	0.4654	0.4145		k_{MU2}	0.6447	0.2206	0.2205	0.1839
	k_{MU3}	0.4518	0.4654	0.4654	0.4339		k_{MU3}	0.6447	0.2206	0.2205	0.1914
	k_{MU4}	0.4518	0.4654	0.4654	0.4627		k_{MU4}	0.6447	0.2206	0.2205	0.2182

3.1 The Discussion of Simulation Results

According to the simulation study, we present the discussion results of this study for all cases of the sample size n , correlation coefficient $\emptyset$ and the standard deviation σ. From Table 1 to Table 16, the performance of the RLS, RL, RAUTPE and SRRE estimators will be discussed for all different cases as follows:

3.1.1 Simulation Results According to The Sample Size

From Table 1 to Table 6, when the value of sample size $(n = 50)$, the EMSE is high for some restricted estimators and the best performance was the SRRE because it has minimum EMSE . While, from Table 7 to Table 16, when $n = 100, 150$, the EMSE is decreased and the best estimator is the SRRE. That means, when the sample size is increased, the EMSE will be decreased and this has been demonstrated by Nejarian [15].

3.1.2 Simulation Results According to Correlation Coefficient $\emptyset$ and the Standard Deviation σ.

1.When ($\emptyset = .85$, $\sigma = 0.1, 1$) , the EMSE is high for all estimators, while when $\sigma = 5 , 15$ the EMSE is starting to be decreased. Still the best estimator is the SRRE.

2.When ($\emptyset = .95$, $\sigma = 0.1,$), the best restricted estimator according to the EMSE is the SRRE which has minimum EMSE.

3. When ($\emptyset = .99$, $\sigma = 0.1,$), the performance of the SRRE is better than of any estimator because it has minimum EMSE . the results are high. Also in case $(\sigma = 1, 5, 15)$, the SRRE is best and the results are better .

4. Numerical Example

In this section, we want to show that, the performance of the RLS, RLE, RAUTPE and the SRRE estimators by using real life data. We are using the data set of the gross national product that applied wildly as used by Akdeniz [16] and Gruber [17]. The goal of using this example is to compare the restricted estimators that are given in this study and also to determine which of these estimators has good statistical properties compared to others. the mse criterion is used to compare of the RLS , RLE , RAUTPE and the SRRE estimators . So that, the mse of the RLE, RAUTPE and SRRE are given in Eq(13), (17) and (22) respectively. According to Najarian [16], the values of *R* and *r* are respectively given as follows:

$R = [1\ 1\ 1\ 1\ ; 0\ 1\ 3\ 1\ ]$, $r = [1.2170\ \ 1.0904]$

Table 17: The scalar mean square error of the RLS, RLE, RAUTPE and SRRE estimators for different estimated ridge parameter k

k	β_{RLS}	β_{rd}	β_{RAUTPE}	β_{SRRE}
0.0161	135.7475	112.3981	23.3420	98.7085
0.050	135.7475	112.3981	23.2073	48.7054
0.10	135.7475	112.3981	23.1207	30.4944
0.15	135.7475	112.3981	23.0700	22.0940
0.20	135.7475	112.3981	23.0377	17.3355
0.25	135.7475	112.3981	23.0150	14.3978
0.30	135.7475	112.3981	22.9985	12.2229
0.35	135.7475	112.3981	22.9856	10.7027

From Table 17, we can observe that, when $k = 0.050, 0.10$ the RAUTPE estimator is better than of any estimator. While, when $k = 0.15 , 0.20 , 0.25, 0.30, 0.35$ the SRRE estimator is better than of any estimator .That means, when the estimated of ridge parameter k are increased the performance of the SRRE estimator is the best and we can observe that of the figures 1, 2 and 3

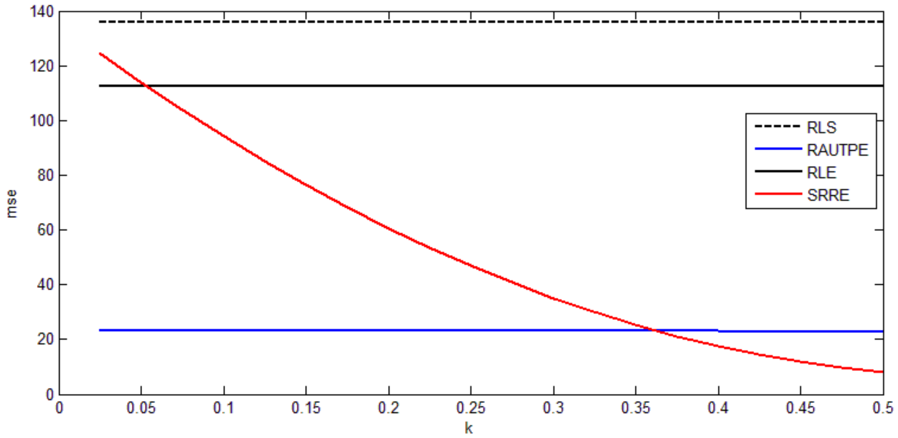

Fig. 1: The mse of RLS, RL, RAUTPE and SRRE estimators for different estimated ridge parameter k

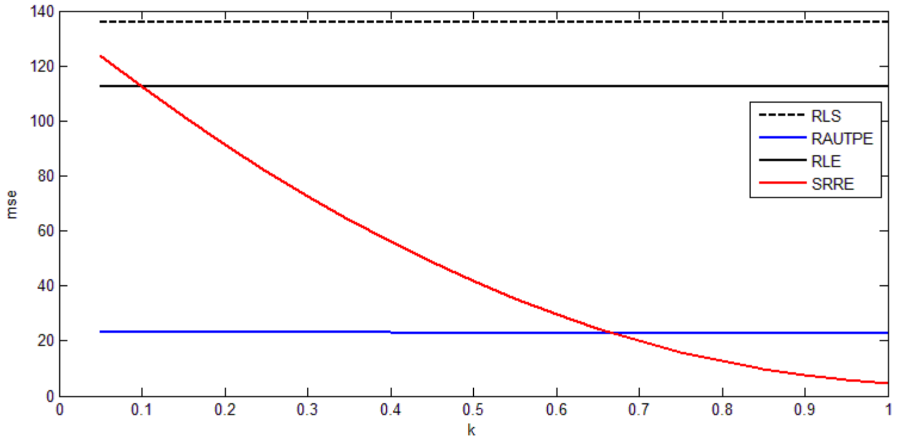

Fig. 2: The mse of RLS, RL, RAUTPE and SRRE estimators for different estimated ridge parameter k

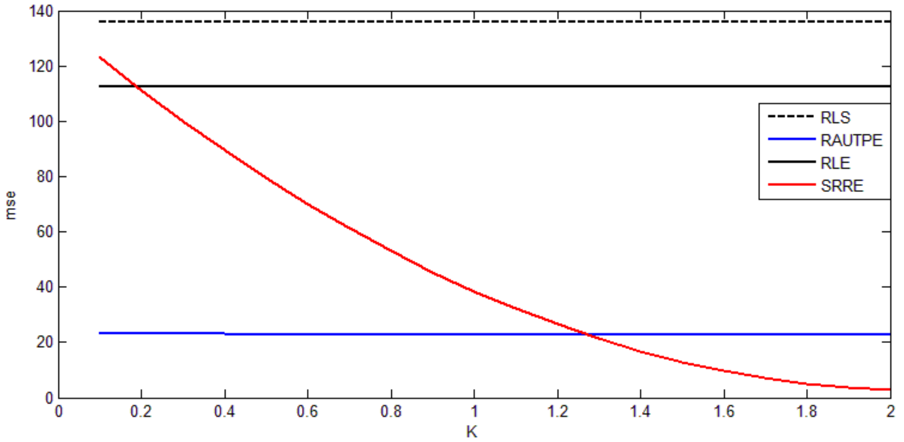

Fig. 3: The mse of RLS, RL, RAUTPE and SRRE estimators for different estimated ridge parameter k

5 . Conclusion

The researchers tried to take advantage from prior information of the parameters through introduced the restricted biased estimator. The purpose of this study is to find out the performance of these estimators. According to the simulation study, we observed that, when the variance is high, the SRRE estimator has minimum mean square error MSE comparing of other restricted estimators, that mean the SRRE estimator has good properties, also, we were able to illustrate this through a numerical example and some figures.

References

[1] Bader. A. M and Mustafa I. N (2021). New Shrinkage Restricted Estimator For Restricted Linear Regression Model .AIP Conference Proceedings (Under Publication).

[2] S. Kaçıranlar, S. Sakallıo˘glu, F. Akdeniz, G.P.H. Styan, and H.J. Werner (1999), A new biased estimator in linear regression and a detailed analysis of the widely analyzed dataset on Portland Cement. Sankhya: Indian J. Stat. B 61(3), pp. 443–459

[3] Hua Huang, Jibo & Wende (2016). On the restricted almost unbiased two parameter estimator in linear regression model, Communications in Statistics- Theory and Methods

[4] Özkale, M. R., and S. Kaciranlar, (2007): The restricted and unrestricted two-parameter estimators. Communications in Statistics—Theory and Methods, 36, 2707-2725.

[5] Hoerl, A. E. & Kennard, R. W. (1970). Ridge regression: Biased estimation for non- orthogonal problems. Technometrics, 12(1), 55-67.

[6] Hoerl, A. E., Kennard, R. W., & Baldwin, K. F. (1975). Ridge regression: Some simulations. Communications in Statistics, 4(2), 105-123.

[7] Lawless, J. F. & Wang, P. (1976). A simulation study of ridge and other regression estimators. Communications in Statistics – Theory and Methods, 5(4), 307-323

[8] Hocking, R. R., Speed, F. M., & Lynn, M. J. (1976). A class of biased estimators in linear regression. Technometrics, 18(4), 55-67.

[9] Nomura, M. (1988). On the almost unbiased ridge regression estimation. Communication in Statistics – Simulation and Computation, 17(3), 729-743.

[10] Kibria, B. M., & Banik, S. (2016). Some ridge regression estimators and their performances. Journal of Modern Applied Statistical Methods, 15(1), 12.

[11] Khalaf, G. & Shukur, G. (2005). Choosing ridge parameters for regression problems. Communications in Statistics – Theory and Methods, 34(5), 1177-1182.

[12] Alkhamisi, M., Khalaf, G., & Shukur, G. (2006). Some modifications for choosing ridge parameters. Communications in Statistics – Theory and Methods, 35(11), 2005-2020.

[13] Kibria, B. M. G. (2003). Performance of some new ridge regression estimators. Communications in Statistics – Simulation and Computation, 32(2), 419-435

[14] M.N Lattef, M .I ALheety (2020). Study of Some Kinds of Ridge Regression Estimators in Linear Regression Model. Tikrit Journal of Pure Since,[5.1] v.25,n.1 ,pp. 130-142.

[15] Najarian, S., Arashi, M. and Kibria, B. M. G. (2013). A Simulation Study on Some Restricted Ridge Regression Estimators. Comm. Statist. Sim. Comp., 42, 871-879

[16] Akdeniz, F., Erol, H. (2003). Mean squared error matrix comparisons of some biased estimators in linear regression. Communications in Statistics Theory and Methods, 32:2389–2413

[17] Gruber, M. (2017). Improving Efficiency by Shrinkage: The James--Stein and Ridge Regression Estimators. Routledge.

دراسة المحاكاة لبعض المقدرات المقيدة في نموذج الانحدار الخطي المقيد

بدر عبود محمد ، مصطفى اسماعيل نايف

قسم الرياضيات ، كلية التربية للعلوم الصرفة ، جامعة الانبار ، الانبار ، العراق

الملخص

عندما تكون مشكلة التداخل الخطي في نموذج الانحدار الخطي موجودة فان نتائج مقدر المربعات الصغرى المقيد تكون غير مستقرة لذلك فان الكثير من الباحثين استخدموا التقدير المقيد لتحسين كفاءة مقدر المربعات الصغرى المقيد في هذا الفصل قمنا بعمل محاكاة لبعض المقدرات المقيدة الغرض من هذه الدراسة هو معرفة اداء هذه المقدرات . وفق هذه الدراسة تبين ان المقدر the Shrinkage restricted ridge regression estimator (SRRE) المقترح بواسطة بدر و الهيتي(2021) يمتلك خصائص جيدة مقارنة مع مقدر المربعات الصغرى المقيد وبعض المقدرات الاخرى وتم توضيح ذلك من خلال المثال العددي.