



ON (sub- super) asymptotic martingales

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ABSTRACT

In this paper we introduce a new class of definitions (sub - super) asymptotic martingale through the concept of asymptotic martingale. we investigate and prove some properties of asymptotic martingale and (sub - super) asymptotic martingale .

1- Introduction

Theory of asymptotic martingales(amart) has been developed and extensively studied in recent years by Bellow[5], Edgar and Sucheston [7], Chacon and Sucheston[6]. These authors are the first to believe that the notion merits a name: asymptotic martingale. A systematic presentation of amart theory paralleling the martingale theory, including for the first time the optional sampling theorem, the Riesz decomposition ,the descending and the parameter cases ,was given by edgar.

among other. It was show that every real valued amart and every vector- valued uniform amart has aRies decomposition.

the amart combines several useful properties of the martingale, submartingale , supermartingale. Thus the class of martingales is closed under linear combinations ,the class of supermartingales under infimum, the class of submartingales under supremum , but the class of amart is closed under all three operations[2].

In (1975) R. V .chacon and L. Sucheston, reduced on convergence of vector asymptotic martingale [8].

.we recall that the definition of asymptotic martingale, A sequence $\{X_n, \mathcal{F}_n, n \geq 1\}$ **if and only if** for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T, t, \sigma \geq t_0$ a.s. We have

$$\| \int X_t dp - \int X_\sigma dp \| < \varepsilon . [3] .$$

our aim in this paper is to study and proves some properties of the asymptotic martingale and (sub,super) asymptotic martingale and every martingale is asymptotic martingale .

2 – (sub - super)asymptotic martingale.

In this section we introduce a new class of definitions(sub - super)asymptotic martingale and prove some properties of asymptotic martingale and (sub - super) asymptotic martingale .

Definition 2.1

A sequence $\{X_n, \mathcal{F}_n, n \geq 1\}$ is called a sub asymptotic martingale (for short - subamart) **if and only if** for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T, t, \sigma \geq t_0$ a.s.

We have

$$\| \int X_\sigma dp - \int X_t dp \| \leq \varepsilon .$$

Definition 2.2

A sequence $\{X_n, \mathcal{F}_n, n \geq 1\}$ is called a super asymptotic martingale (for short - superamart) **if and only if** for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T, t, \sigma \geq t_0$ a.s.

We have

$$\| \int X_t dp - \int X_\sigma dp \| \leq \varepsilon .$$

Remark 2 .3

1-Martingale \Rightarrow asymptotic martingale \Rightarrow super asymptotic martingale.

2- sub asymptotic martingale \Leftrightarrow super asymptotic martingale.

Example 2.4

Let X_n be integrable random variable and $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots$ be a filter on probability space (Ω, \mathcal{F}, P) , $E|X| < \infty$ and define $X_n = a/2^n$, then $\{X_n, \mathcal{F}_n\}$ is asymptotic martingale.

Theorem 2.5

If $\{X_n, \mathcal{F}_n, n \geq 1\}$ and $\{Y_n, \mathcal{F}_n, n \geq 1\}$ are asymptotic martingale, then $\{X_n + Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale

With corresponding statements for subasymptotic martingale and superasymptotic martingale.

Proof

Since $\{X_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale, for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T, t, \sigma \geq t_0$,

$$\| \int X_t dp - \int X_\sigma dp \| < 1/2\varepsilon \dots (2.5.1)$$

since $\{Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale, for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T, t, \sigma \geq t_0$,

$$\| \int Y_t dp - \int Y_\sigma dp \| < 1/2\varepsilon \dots (2.5.2)$$

We want to prove that

$$\| \int (X_t + Y_t) dp - \int (X_\sigma + Y_\sigma) dp \| < \varepsilon$$

We have from (2.5.1) and (2.5.2) we get

$$\| \int (X_t + Y_t) dp - \int (X_\sigma + Y_\sigma) dp \| = \| \int X_t dp - \int X_\sigma dp + \int Y_t dp - \int Y_\sigma dp \| \leq \| \int X_t dp - \int X_\sigma dp \| + \| \int Y_t dp - \int Y_\sigma dp \| < 1/2\varepsilon + 1/2\varepsilon = \varepsilon$$

Thus $\{X_n + Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale.

Theorem 2.6

If $\{X_n, \mathcal{F}_n, n \geq 1\}$ and $\{Y_n, \mathcal{F}_n, n \geq 1\}$ are asymptotic martingale, then $\{X_n - Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale

With corresponding statements for subasymptotic martingale and superasymptotic martingale.

Proof

since $\{X_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale, for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T, t, \sigma \geq t_0$,

$$\| \int X_t dp - \int X_\sigma dp \| < 1/2\varepsilon \dots (2.6.1)$$

since $\{Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale,

for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T, t, \sigma \geq t_0$,

$$\| \int Y_t dp - \int Y_\sigma dp \| < 1/2\varepsilon \dots (2.6.2)$$

We want to prove that

$$\| \int (X_t - Y_t) dp - \int (X_\sigma - Y_\sigma) dp \| < \varepsilon$$

We have from (2.6.1) and (2.6.2) we get

$$\| \int (X_t - Y_t) dp - \int (X_\sigma - Y_\sigma) dp \| = \| \int X_t dp - \int X_\sigma dp - \int Y_t dp + \int Y_\sigma dp \| \leq \| \int X_t dp - \int X_\sigma dp \| + \| \int Y_t dp - \int Y_\sigma dp \| < 1/2\varepsilon + 1/2\varepsilon = \varepsilon$$

Thus $\{X_n - Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale

Theorem 2.7

If $\{aX_n, \mathcal{F}_n, n \geq 1\}$ and $\{bY_n, \mathcal{F}_n, n \geq 1\}$ are asymptotic martingale, then $\{aX_n + bY_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale

With corresponding statements for subasymptotic martingale and superasymptotic martingale.

Proof

Since $\{aX_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale, for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T, t, \sigma \geq t_0$,

$$\| \int aX_t dp - \int aX_\sigma dp \| < 1/2\varepsilon \dots (2.7.1)$$

since $\{bY_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale,

for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T, t, \sigma \geq t_0$,

$$\| \int bY_t dp - \int bY_\sigma dp \| < 1/2\varepsilon \dots (2.7.2)$$

We want to prove that $\| \int (aX_t + bY_t) dp - \int (aX_\sigma + bY_\sigma) dp \| < \varepsilon$

We have from (2.7.1) and (2.7.2) we get

$$\| \int (aX_t + bY_t) dp - \int (aX_\sigma + bY_\sigma) dp \| = \| \int aX_t dp - \int aX_\sigma dp + \int bY_t dp - \int bY_\sigma dp \| \leq \| \int aX_t dp - \int aX_\sigma dp \| + \| \int bY_t dp - \int bY_\sigma dp \| < 1/2\varepsilon + 1/2\varepsilon = \varepsilon$$

Thus $\{aX_n + bY_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale

Theorem 2.8

If $\{Y_n, \mathcal{F}_n\}$ is a sub asymptotic martingale, then $\{\alpha Y_n, \mathcal{F}_n\}$, for $\alpha < 0$ is a super asymptotic martingale $\{\alpha Y_n, \mathcal{F}_n\}$, for $\alpha > 0$ is a sub asymptotic martingale.

Proof

for $\alpha < 0$, since $\{Y_n, \mathcal{F}_n\}$ is a sub asymptotic martingale,

for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T, t, \sigma \geq t_0$,

$$\| \int Y_\sigma dp - \int Y_t dp \| \leq \varepsilon_1.$$

That is implies

$$\| \alpha \| \int Y_\sigma dp - \int Y_t dp \| = \| \int \alpha Y_t dp - \int \alpha Y_\sigma dp \| \leq |\alpha| \varepsilon_1 = \varepsilon$$

That is $\{\alpha Y_n, \mathcal{F}_n\}$ is a super asymptotic martingale

for $\alpha > 0$, since $\{Y_n, \mathcal{F}_n\}$ is a sub asymptotic martingale,

for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T, t, \sigma \geq t_0$,

$$\| \int Y_\sigma dp - \int Y_t dp \| \leq \varepsilon_1.$$

That is implies

$$\| \alpha \| \int Y_\sigma dp - \int Y_t dp \| = \| \int \alpha Y_\sigma dp - \int \alpha Y_t dp \| \leq |\alpha| \varepsilon_1 = \varepsilon$$

That is $\{\alpha Y_n, \mathcal{F}_n\}$ is a sub asymptotic martingale.

Theorem 2.9

If $\{X_{1,n}, \mathcal{F}_n\}, \dots, \{X_{m,n}, \mathcal{F}_n\}$ are a super asymptotic martingale, then $\{\min(X_{1,n}, \dots, X_{m,n}), \mathcal{F}_n\}$ is a super asymptotic martingale.

Proof

We want to prove

$$\| \int \min(X_{t1,n}, \dots, X_{tm,n}) dp - \int \min(X_{\sigma1,n}, \dots, X_{\sigma m,n}) dp \| \leq \varepsilon$$

Since $\{X_{1,n}, \mathcal{F}_n\}, \dots, \{X_{m,n}, \mathcal{F}_n\}$ are a super asymptotic martingale, for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T, t, \sigma \geq t_0$,

$$\| \int X_{t1,n} dp - \int X_{\sigma1,n} dp \| \leq \varepsilon, \dots,$$

$$\| \int X_{tm,n} dp - \int X_{\sigma m,n} dp \| \leq \varepsilon$$

Consider

$$\| \int \min(X_{t1,n}, \dots, X_{tm,n}) dp - \int \min(X_{\sigma1,n}, \dots, X_{\sigma m,n}) dp \| \leq \| \int X_{t1,n} dp - \int X_{\sigma1,n} dp \| \leq \varepsilon \dots (2.9.1)$$

$$\| \int \min(X_{t1,n}, \dots, X_{tm,n}) dp - \int \min(X_{\sigma1,n}, \dots, X_{\sigma m,n}) dp \| \leq \| \int X_{t2,n} dp - \int X_{\sigma2,n} dp \| \leq \varepsilon \dots \dots (2.9.2)$$

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$$\| \int \min(X_{t1,n}, \dots, X_{tm,n}) dp - \int \min(X_{\sigma1,n}, \dots, X_{\sigma m,n}) dp \| \leq \| \int X_{tm,n} dp - \int X_{\sigma m,n} dp \| \leq \varepsilon \dots \dots (2.9. m)$$

From (2.9.1), ..., (2.9. m) we obtain

$$\| \int \min(X_{t1,n}, \dots, X_{tm,n}) dp - \int \min(X_{\sigma1,n}, \dots, X_{\sigma m,n}) dp \| \leq \varepsilon$$

Theorem 2.10

If $\{X_{1,n}, \mathcal{F}_n\}, \dots, \{X_{m,n}, \mathcal{F}_n\}$ are a sub asymptotic martingale, then $\{\min(X_{1,n}, \dots, X_{m,n}), \mathcal{F}_n\}$ is a sub asymptotic martingale.

Proof

We want to prove

$$\| \int \min(X_{\sigma1,n}, \dots, X_{\sigma m,n}) dp - \int \min(X_{t1,n}, \dots, X_{tm,n}) dp \| \leq \varepsilon$$

Since $\{X_{1,n}, \mathcal{F}_n\}, \dots, \{X_{m,n}, \mathcal{F}_n\}$ are a sub asymptotic martingale for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T, t, \sigma \geq t_0$,

$$\| \int X_{\sigma1,n} dp - \int X_{t1,n} dp \| \leq \varepsilon \dots \dots$$

$$\| \int X_{\sigma m,n} dp - \int X_{tm,n} dp \| \leq \varepsilon$$

Consider

$$\| \int \min(X_{\sigma1,n}, \dots, X_{\sigma m,n}) dp - \int \min(X_{t1,n}, \dots, X_{tm,n}) dp \| \leq \| \int X_{\sigma1,n} dp - \int X_{t1,n} dp \| \leq \varepsilon \dots \dots (2.10.1)$$

$$\| \int \min(X_{\sigma1,n}, \dots, X_{\sigma m,n}) dp - \int \min(X_{t1,n}, \dots, X_{tm,n}) dp \| \leq \| \int X_{\sigma2,n} dp - \int X_{t2,n} dp \| \leq \varepsilon \dots \dots (2.10.2)$$

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$$\| \int \min(X_{\sigma1,n}, \dots, X_{\sigma m,n}) dp - \int \min(X_{t1,n}, \dots, X_{tm,n}) dp \| \leq \| \int X_{\sigma m,n} dp - \int X_{tm,n} dp \| \leq \varepsilon \dots \dots (2.10. m)$$

From (2.10.1), ..., (2.10. m) we obtain

$$\| \int \min(X_{\sigma1,n}, \dots, X_{\sigma m,n}) dp - \int \min(X_{t1,n}, \dots, X_{tm,n}) dp \| \leq \varepsilon$$

Theorem 2.11

If $\{X_n, \mathcal{F}_n, n \geq 1\}$ and $\{Y_n, \mathcal{F}_n, n \geq 1\}$ are asymptotic martingale and $|X_t| = |Y_\sigma| = 1$, then $\{X_n Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale

With corresponding statements for subasymptotic martingale and superasymptotic martingale.

Proof

Since $\{X_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale, for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T, t, \sigma \geq t_0$, then $\| \int X_t dp - \int X_\sigma dp \| < 1/2\varepsilon$

Since $\{Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale, for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T, t, \sigma \geq t_0$, then $\| \int Y_t dp - \int Y_\sigma dp \| < 1/2\varepsilon$

We have to prove

$$\| \int X_t Y_t dp - \int X_\sigma Y_\sigma dp \| < \varepsilon$$

$$\| \int X_t Y_t dp - \int X_\sigma Y_\sigma dp \| = \| \int X_t Y_t dp - \int X_t Y_\sigma dp + \int X_t Y_\sigma dp - \int X_\sigma Y_\sigma dp \|$$

$$= \| X_t (\int Y_t dp - \int Y_\sigma dp) + Y_\sigma (\int X_t dp - \int X_\sigma dp) \|$$

$$\leq |X_t| \| \int Y_t dp - \int Y_\sigma dp \| + |Y_\sigma| \| \int X_t dp - \int X_\sigma dp \|$$

$$< |X_t| 1/2\varepsilon + |Y_\sigma| 1/2\varepsilon = 1/2\varepsilon (|X_t| + |Y_\sigma|)$$

$$= 1/2\varepsilon (2) = \varepsilon$$

Thus $\{X_n Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale.

Example 2.12

Let $\{X_n, n \geq 1\}$ be an asymptotic martingale for increasing family $\{\mathcal{F}_n, n \geq 1\}$ of σ -field. Then for all $n \geq 1, Y_n = \sum_{k=1}^n X_k$ is an asymptotic martingale for increasing family $\{\mathcal{F}_n, n \geq 1\}$ of σ -field.

solution

for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T,$

$t, \sigma \geq t_0$, Then

$$\| \int Y_t dp - \int Y_\sigma dp \| = \| \int \sum_{k=1}^t X_k dp - \int \sum_{k=1}^\sigma X_k dp \| = \| \int (X_1 + X_2 + \dots + X_t) dp - \int (X_1 + X_2 + \dots + X_\sigma) dp \|$$

$$= \| \int X_1 dp + \int X_2 dp + \dots + \int X_t dp - \int X_1 dp - \int X_2 dp - \dots - \int X_\sigma dp \|$$

$$= \| \int X_t dp - \int X_\sigma dp \|$$

Since $\{X_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale

$$\| \int X_t dp - \int X_\sigma dp \| < \varepsilon$$

Thus $\{Y_n, \mathcal{F}_n, n \geq 1\}$ is an asymptotic martingale.

Lemma 2.13[2]

Let $\{X_n, n \geq 1\}$ be an asymptotic martingale for $\{\mathcal{F}_n, n \geq 1\}$. Then $(\int Z_t)_{t \in T}$ is bounded.

proposition 2.14[2]

Let $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be sequences adapted to $\{\mathcal{F}_n, n \geq 1\}$. assume in addition that they are L^1 -bounded. Then if $(\int X_t)_{t \in T}$ and $(\int Y_t)_{t \in T}$ are bounded then $(\int X_t \vee Y_t)_{t \in T}$ and $(\int X_t \wedge Y_t)_{t \in T}$ are bounded.

proposition 2.15[2]

If $\{X_n, \mathcal{F}_n, n \geq 1\}$ and $\{Y_n, \mathcal{F}_n, n \geq 1\}$ are asymptotic martingale, then

$\{X_n \vee Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale.

Proof

We have to prove that $X_n \vee Y_n$ is asymptotic martingale. Write $Z_n = X_n \vee Y_n$. By lemma(2.13), $(\int X_t)$ and $(\int Y_t)$ are bounded. By proposition(2.14), $(\int Z_t)_{t \in T}$ is bounded. let $\varepsilon > 0$ be given we can choose $t_0 \in T$ such if $t, \sigma \geq t_0$, then

$$\| \int X_\sigma - \int X_t \| < \varepsilon, \| \int Y_\sigma - \int Y_t \| < \varepsilon \dots \dots (2.15.1)$$

Since, $(\int Z_t)_{t \in T}$ is bounded. we can choose $t_1 \geq t_0$ such that if $\sigma \geq t_0$, then $\int Z_\sigma \leq \int Z_{t_1} + \varepsilon \dots \dots (2.15.2)$

Now given any bounded stopping time $\sigma \geq t_1$, let $\{X_{t_1} < Y_{t_1}\}$ and define $\sigma_1 \in T$ by

$$\sigma_1 = t_1 \text{ on } A$$

$$= \sigma \text{ on } A^c.$$

$$\text{Then } \int X_{t_1} = \int A^c Z_{t_1} + \int A X_{t_1} \dots \dots (2.15.3)$$

$$\int X_{\sigma_1} = \int A^c X_\sigma + \int A X_{t_1} \dots \dots (2.15.4)$$

Subtracting(1.15.4) from (1.15.3), then using (1.15.1), we have

$$\int A^c Z_{t_1} = \int A^c X_\sigma + \int X_{t_1} - \int X_{\sigma_1} \leq \int A^c Z_\sigma + \varepsilon \dots (2.15.5)$$

$$\text{Again } \int Y_{t_1} = \int A Z_{t_1} + \int A^c Y_{t_1} \dots \dots (2.15.3')$$

$$\int Y_{\sigma_1} = \int A Y_{\sigma} + A^c Y_{t_1} \dots \dots \dots (2.15.4')$$

Subtracting (2.15.4') from (2.15.3'), then using (2.15.1), we have

$$\int A Z_{t_1} = \int A Y_{\sigma} + \int Y_{t_1} - \int Y_{\sigma_1} \leq \int A Z_{\sigma} + \varepsilon \dots \dots (2.15.5')$$

$$\int Z_{t_1} \leq \int Z_{\sigma} + 2\varepsilon$$

This, together with (2.15.2), yields

$$|\int Z_{\sigma} - \int Z_t| < 2\varepsilon \dots \dots \dots (2.15.6)$$

This shows that the net $(\int Z_t)_{t \in T}$ is Cauchy, hence convergent.

Definition 2.16[3]

Let (Ω, \mathcal{F}, P) be a probability space, $\{X_1, X_2, \dots\}$ a sequence of integrable random variable on (Ω, \mathcal{F}, P) and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$ an increasing sequence of sub σ -field of \mathcal{F} , X_n is assumed \mathcal{F}_n -measurable that is $X_n : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

The sequence $\{X_n, \mathcal{F}_n\}$ is said to be a Martingale **if and only if** for all $n=1, 2, \dots, E[X_{n+1} | \mathcal{F}_n] = X_n$ a.e and a sub martingale **if and only if** $E[X_{n+1} | \mathcal{F}_n] \geq X_n$ a.s.; and supermartingale **if and only if** $E[X_{n+1} | \mathcal{F}_n] \leq X_n$ a.e.

Example 2.17

Let B_t^2 be a Brownian motion we compute $E[B_{t+s}^2 - t | \mathcal{F}_s]$ for $t \geq s$.

$$E[B_{t+s}^2 - t | \mathcal{F}_s] = E[(B_t - B_s + B_s)^2 - t | \mathcal{F}_s] = E[(B_t - B_s + B_s)^2 | \mathcal{F}_s] - t$$

$$E[(B_t - B_s)^2 | \mathcal{F}_s] + E[B_s^2 | \mathcal{F}_s] + 2E[(B_t - B_s)B_s | \mathcal{F}_s] - t$$

Using independence, $E[(B_t - B_s)^2 | \mathcal{F}_s] = E[(B_t - B_s)^2] = t - s$

Of course $E[B_s^2 | \mathcal{F}_s] = B_s^2$

In the last term we use property $\{y \in \mathcal{F}, E[XY] < \infty\}$, then $E[(B_t - B_s)B_s | \mathcal{F}_s] = B_s E[(B_t - B_s) | \mathcal{F}_s] = B_s E[B_t - B_s] = 0$

Since $B_t - B_s$ is independence of \mathcal{F}_s

$$\text{Thus } E[B_{t+s}^2 - t | \mathcal{F}_s] = B_s^2 + t - s - t = B_s^2 - s$$

Thus $B_{t+s}^2 - t$ is a martingale.

Theorem 2.18[2]

Every martingale is asymptotic martingale.

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Proof

The adapted sequence $(X_n)_{n \in \mathbb{D}}$ is defined so that martingale **if and only if**

$$\int |X_n| < \infty \text{ for all } n \text{ and } E[X_n | \mathcal{F}_m] = X_m \text{ for all } n, m \in \mathbb{D} \text{ with } n \geq m, \text{ in particular } \int X_n = \int X_m$$

If $t \in \tau$, choose $n \in \mathbb{D}$ with $n \geq t$.

Then $\int X_t = \int \sum_{k=t}^n X_k = \sum_{k=t}^n \int X_k = \int X_n$. Thus $(\int X_t)$ is constant, the sequence $(X_n)_{n \in \mathbb{D}}$ is asymptotic martingale.

Theorem 2.19

Let $(X_n, n \geq 1)$ be a martingale for increasing family $(\mathcal{F}_n, n \geq 1)$ of σ -field. Let $(\mathcal{G}_n, n \geq 1)$ be another increasing family of σ -field with $\mathcal{G}_n \subseteq \mathcal{F}_n$ for all $n \in \mathbb{N}$. Then $Y_n = E(X_n | \mathcal{G}_n)$ is an asymptotic martingale for $(\mathcal{G}_n, n \geq 1)$ and not martingale.

proof

we want to prove $(Y_n, \mathcal{G}_n, n \geq 1)$ is asymptotic martingale,

for every $\varepsilon > 0$, there exist $t_0 \in T$ such that for every $t, \sigma \in T, t, \sigma \geq t_0$,

$$\|\int Y_t dp - \int Y_{\sigma} dp\| = \|\int E(X_t | \mathcal{G}_t) dp - \int E(X_{\sigma} | \mathcal{G}_{\sigma}) dp\| = \|\int E(E(X_{t+1} | \mathcal{F}_t) | \mathcal{G}_t) dp - \int E(E(X_{\sigma+1} | \mathcal{F}_{\sigma}) | \mathcal{G}_{\sigma}) dp\|$$

Since $\mathcal{G}_n \subseteq \mathcal{F}_n$

$$\|\int E(X_{t+1} | \mathcal{G}_t) dp - \int E(X_{\sigma+1} | \mathcal{G}_{\sigma}) dp\| = \|\int X_t dp - \int X_{\sigma} dp\| \text{ by theorem (1.17)}$$

$$\|\int X_t dp - \int X_{\sigma} dp\| < \varepsilon$$

Thus $(Y_n, \mathcal{G}_n, n \geq 1)$ is asymptotic martingale.

We want to prove $(Y_n, \mathcal{G}_n, n \geq 1)$ is not martingale

$$E(Y_{n+1} | \mathcal{G}_n) = E(E(X_{n+1} | \mathcal{G}_{n+1}) | \mathcal{G}_n) = E(X_{n+1} | \mathcal{G}_n) = X_n \neq Y_n$$

Theorem 2.20[4]

If $\{X_n, \mathcal{F}_n, n \geq 1\}$ and $\{Y_n, \mathcal{F}_n, n \geq 1\}$ are martingale, then $\{X_n Y_n, \mathcal{F}_n, n \geq 1\}$ is martingale.

Theorem 2.21

If $\{X_n, \mathcal{F}_n, n \geq 1\}$ and $\{Y_n, \mathcal{F}_n, n \geq 1\}$ are martingale, then $\{X_n Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale.

Proof

By **theorem (2.18)** and **theorem (2.20)**, then $\{X_n Y_n, \mathcal{F}_n, n \geq 1\}$ is asymptotic martingale.

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حول المارتنكل المقارب الجزئي والخاص

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الملخص

في هذا البحث قدمنا صنف جديد من مفاهيم وتعريف المارتنكل المقارب بنوعيه الجزئي والخاص وبرهنا بعض خصائص المارتنكل المقارب الجزئي والخاص.